# Strong reductions in effective randomness

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**Abstract**

We study generalizations of Demuth's Theorem, which states that the image of a Martin-Löf random real under a *tt*-reduction is either computable or Turing equivalent to a Martin-Löf random real. We show that Demuth's Theorem holds for Schnorr randomness and computable randomness (answering a question of Franklin), but that it cannot be strengthened by replacing the Turing equivalence in the statement of the theorem with *wtt*-equivalence. We also provide some additional results about the Turing and *tt*-degrees of reals that are random with respect to some computable measure.

*Key words:* Algorithmic randomness, Demuth's Theorem, computable analysis.

## 1 Introduction

The main topic of this paper is a theorem of Oswald Demuth's concerning effective randomness. Demuth's Theorem is often stated as follows. Given a Martin-Löf random real $x$, every non-computable real $y$ that $x$ *tt*-computes is in turn *Turing* equivalent to a Martin-Löf random real $z$. Demuth's original result (in the paper [Dem88]) is written using slightly outdated terminology (Martin-Löf random reals are for example called "non-approximable"), and has sometimes been mistranslated in modern parlance. For example, a stronger version has repeatedly appeared in circulated drafts, talks, and even in some published papers, which asserts that one can even require that $y$ is *wtt*-equivalent to $z$. This is for example the version given in [Fra08], where the author further asks:

(a) whether $z$ can further be required to be *tt*-equivalent to $y$; and

(b) whether Demuth's Theorem also holds for Schnorr randomness.

In attempting to answer question (a), we realized that even the "*wtt*-version" of Demuth's Theorem was in fact false, as we will show below. We will also answer question (b) positively.

The rest of the paper is organized as follows. In the first half of the paper, we study the generalization of Demuth's Theorem to different notions of effective randomness. First, in Section 2 we provide the necessary background on computable probability measures and their connections to strong reductions, leading to a proof of Demuth's Theorem in Section 3. Then, we show in Section 4 that the analogue of Demuth's Theorem holds for other notions of randomness, namely computable randomness and Schnorr randomness, even though the proof requires some additional effort. In Section 5, we study the "*wtt*-analogue" of Demuth's Theorem and show that it fails for all notions of randomness considered in the paper. The last section, Section 6, is a general study of Turing degrees of reals that are random with respect to some computable probability measure. For completeness, the proofs of previously known results as well as technical lemmas needed in our discussion are given in a separate appendix.

We assume that the reader is familiar with the basics of computability theory: computable functions, partial computable functions, computably enumerable sets, Turing functionals, Turing degrees, the Turing jump, and so on (see, for instance, [Soa87]), as well as the basics of effective randomness (otherwise, we refer the reader to [DH10] or [Nie09]). Let us fix some notation and terminology. We denote by $2^\omega$ the set of infinite binary sequences, also known as Cantor space. We denote the set of finite strings by $2^{<\omega}$ and the empty string by $\varnothing$. $\mathbb{Q}_2$ is the set of dyadic rationals, i.e., multiples of a negative power of 2. Given $x \in 2^\omega$ and an integer $n$, $x{\upharpoonright}n$ is the string that consists of the first $n$ bits of $x$, and $x(n)$ is the $(n+1)$st of $x$ (so that $x(0)$ is the first bit of $x$). If $\sigma$ is a string, and $x$ is either a string or an infinite sequence, then $\sigma \preceq x$ means that $\sigma$ is a prefix of $x$. Given a string $\sigma$, the *cylinder* $[\![\sigma]\!]$ is the set of elements of $2^\omega$ having $\sigma$ as a prefix. Moreover, given $S \subseteq 2^{<\omega}$, $[\![S]\!]$ is defined to be the set $\bigcup_{\sigma \in S}[\![\sigma]\!]$. An element $x \in 2^\omega$ will commonly be identified with the real $\sum_n 2^{-x(n)-1}$, which belongs to $[0,1]$, and elements of $2^\omega$ will sometimes be referred to as *reals*. Note that this correspondence is not one-to-one, but it becomes one-to-one if we remove dyadic rationals. We can define an ordering $\leq$ on $2^\omega$ in terms of the standard ordering $\leq$ on [0,1], so that given $x, y \in 2^\omega$, $x \leq y$ if and only if $\sum_n 2^{-x(n)-1} \leq \sum_n 2^{-y(n)-1}$ (where $\leq$ should not be confused with $\preceq$).

## 2 Background

Although Demuth's Theorem at first glance does not appear to involve probability measures on $2^\omega$ (and, in fact, Demuth did not explicitly make use of computable probability measures in the original proof of his theorem), the approach we take here will concern the probability measures induced by strong Turing reductions. The main two types of reductions we will consider are those induced by total Turing functionals (which are also known as *tt*-functionals) and Turing functionals that are almost total, in a sense we will specify shortly. To this end, in this section we will review the relevant material on (computable) measures and Turing functionals, as well as the various notions of effective randomness considered in this paper.

### *2.1 Computable measures*

In this paper, we consider measures on the Cantor space $2^\omega$ or on the interval $[0,1]$. All the measures we consider are Borel probability measures. Therefore, for the sake of concision we will use "measure" in place of "Borel probability measure".

A measure on $2^\omega$ assigns to each Borel set a real in $[0,1]$. It is however sufficient to consider the restriction of probability measures to cylinders. Indeed, Carathéodory's theorem from classical measure theory ensures that a function $\mu$ defined on cylinders which satisfies both $\mu(\llbracket\varnothing\rrbracket) = 1$ and $\mu(\llbracket\sigma\rrbracket) = \mu(\llbracket\sigma 0\rrbracket) + \mu(\llbracket\sigma 1\rrbracket)$ (for all $\sigma$), can be uniquely extended to a probability measure. We can therefore represent measures as functions from strings to reals, where for all $\sigma \in 2^{<\omega}$, $\mu(\sigma)$ is the measure of the cylinder $\llbracket\sigma\rrbracket$. This concise representation also allows us to talk about *computable* probability measures.

**Definition 2.1** *A probability measure $\mu$ on $2^\omega$ is* computable *if $\sigma \mapsto \mu(\sigma)$ is computable as a real-valued function, i.e. if there is a computable function $\widehat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that*

$$|\mu(\sigma) - \widehat{\mu}(\sigma, i)| \leq 2^{-i}$$

*for every $\sigma \in 2^{<\omega}$ and $i \in \omega$. We further say that $\mu$ is* exactly computable *if for all $\sigma$, $\mu(\sigma) \in \mathbb{Q}_2$ and $\sigma \mapsto \mu(\sigma)$ is computable as a function from $2^{<\omega}$ to $\mathbb{Q}_2$.*

In what follows $\lambda$ will refer exclusively to the Lebesgue measure on $2^\omega$, where $\lambda(\sigma) = 2^{-|\sigma|}$ for each $\sigma \in 2^{<\omega}$. For our purposes, it will be useful to identify several different types of measures.

**Definition 2.2** *A computable measure $\mu$ is* positive *if $\mu(\sigma) > 0$ for every $\sigma \in 2^{<\omega}$. Equivalently, $\mu$ is positive if $\mu(\mathcal{U}) > 0$ for every non-empty open set $\mathcal{U}$. Moreover, $\mu$ is* atomic *if there is some real $x \in 2^\omega$ such that $\mu(\{x\}) > 0$. In this case, we call $x$ an* atom *of $\mu$ or a $\mu$-atom. In $\mu$ has no atoms, we call it* atomless. *Given an atomic measure $\mu$, the collection of $\mu$-atoms will be denoted $\mathsf{Atoms}_\mu$.*

The following result of Kautz is very useful for the present study.

**Proposition 2.3 ([Kau91])** *A real $r$ is computable if and only if $r \in \mathsf{Atoms}_\mu$ for some computable measure $\mu$.*

### *2.2 Strong reductions and induced measures*

Strong reductions play a central role in the discussion that follows, and more generally in the study of effective randomness. For $x, y \in 2^\omega$, we say that $x$ is Turing reducible to $y$, denoted $x \leq_T y$, if there is a Turing functional $\Phi$ such that $\Phi(x) = y$. Recall that $\Phi :\subseteq 2^\omega \to 2^\omega$ is a *Turing functional* (or *reduction*) if there exists a c.e. set $\Gamma$ of pairs of strings such that for all $(\sigma_1, \tau_1)$ and $(\sigma_2, \tau_2)$ in $\Gamma$, if $\sigma_2 \succeq \sigma_1$, then $\tau_2 \succeq \tau_1$. Given two reals $x, y$, we write $\Phi(x) = y$ if for all prefixes $\tau$ of $y$ there is a prefix $\sigma$ of $x$ such that $(\sigma, \tau) \in \Gamma$ and we say that $x$ *Turing computes* $y$ via $\Phi$. Note that given a real $x$ there is at most one real $y$ that $x$ Turing computes via $\Phi$. If there is indeed such a $y$, we say that $\Phi$ is *defined on* $x$ (or $x \in \mathrm{dom}(\Phi)$). If $\Phi(x) = y$, the *use* of $\Phi$ on $x$ is the function $f$ such that for each $n$, $f(n)$ is the least value such that $(x{\upharpoonright}f(n), y{\upharpoonright}n) \in \Gamma$.

**Definition 2.4** *Let $\mu$ be a probability measure on $2^\omega$. A Turing functional $\Phi :\subseteq 2^\omega \to 2^\omega$ is*

*(1)* $\mu$-almost total *if $\mu(dom(\Phi)) = 1$;*
*(2) a* truth-table functional *if $\Phi$ is total;*
*(3) a* weak truth-table functional *if there is some computable function $\varphi$ that bounds the use of $\Phi$ on all reals on which $\Phi$ is defined; and*
*(4)* non-decreasing *if for all $x \leq y$, if both $\Phi(x)$ and $\Phi(y)$ are defined, then $\Phi(x) \leq \Phi(y)$.*

It is immediate that every truth-table functional is almost total. Moreover, every truth-table functional is also a weak truth-table functional. Almost total functionals are important for the study of effective randomness, as we can use them to define computable measures.

**Definition 2.5** *Given a $\mu$-almost total functional $\Phi : 2^\omega \to 2^\omega$, the measure* *induced by $\Phi$, denoted $\mu_\Phi$, is defined to be*

$$\mu_\Phi(\mathcal{X}) = \mu(\Phi^{-1}(\mathcal{X}))$$

*for every measurable $\mathcal{X} \subseteq 2^\omega$.*

An easy but very useful result is that $\mu_\Phi$ can be computed if $\mu$ and $\Phi$ are given.

**Lemma 2.6** *Let $\mu$ be a probability measure on $2^\omega$ and a functional $\Phi : 2^\omega \to 2^\omega$, the following hold.*

(1) *If $\mu$ is computable and $\Phi$ is $\mu$-almost total, then $\mu_\Phi$ is computable.*
(2) *If $\mu$ is exactly computable and $\Phi$ is a tt-functional, then $\mu_\Phi$ is exactly computable.*

**PROOF.** See Appendix.

The induced measure $\mu_\Phi$ as defined above shares certain features of the original measure $\mu$ as long as the functional $\Phi$ satisfies some additional condition:

**Lemma 2.7** *Let $\mu$ be a measure on $2^\omega$ and a $\mu$-almost total functional $\Phi : 2^\omega \to 2^\omega$. The following hold:*

(1) *If $\mu$ is atomless and $\Phi$ is one-to-one, then $\mu_\Phi$ is atomless.*
(2) *If $\mu$ is positive and $\Phi$ is onto, then $\mu_\Phi$ is positive.*

**PROOF.** Suppose $\mu$ is atomless and $\Phi$ is one-to-one. Then for all $x$, $\mu_\Phi(\{x\}) = \mu(\Phi^{-1}(\{x\}))$. Since $\Phi$ is one-to-one, $\Phi^{-1}(\{x\})$ is either empty or is a singleton; $\mu$ being atomless, it follows in either case that $\mu(\Phi^{-1}(\{x\})) = 0$.

Suppose now that $\mu$ is positive and $\Phi$ is onto. Let $\mathcal{U}$ be a non-empty open set. Since $\Phi$ is onto (and continuous on its domain), $\Phi^{-1}(\mathcal{U})$ is a non-empty open set (modulo a set of measure 0 on which $\Phi$ is not defined), and therefore has positive $\mu$-measure.

### *2.3 Notions of randomness*

Although there are many notions of effective randomness for elements of $2^\omega$, we restrict our attention to three of the most important notions: Martin-Löf randomness, Schnorr randomness, and computable randomness. Each of these notions can be defined in one of several ways, but we will restrict our attention in this paper to the measure-theoretic formulations of Martin-Löf randomness and Schnorr randomness and the formulation of computable randomness in terms of certain computable betting strategies known as martingales. For more details, see [DH10] or [Nie09].

**Definition 2.8** *Given a computable measure $\mu$, a* $\mu$-Martin-Löf test *is a uniformly computable sequence $\{\mathcal{U}_i\}_{i\in\omega}$ of effectively open classes in $2^\omega$ such that $\mu(\mathcal{U}_i) \leq 2^{-i}$ for every $i \in \omega$. Further, a real $x$ is* $\mu$-Martin-Löf random *if for every $\mu$-Martin-Löf test $\{\mathcal{U}_i\}_{i\in\omega}$, we have $x \notin \bigcap_{i\in\omega}\mathcal{U}_i$. The collection of $\mu$-Martin-Löf random reals will be written as $\mathsf{MLR}_\mu$.*

As is well-known, for every computable measure $\mu$, there is a universal $\mu$-Martin-Löf test.

**Proposition 2.9** *For every computable measure $\mu$, there is a Martin-Löf test $\{\widehat{\mathcal{U}}_i\}_{i\in\omega}$ such that $x \in \mathsf{MLR}_\mu$ if and only if $x \notin \bigcap_{i\in\omega}\widehat{\mathcal{U}}_i$.*

The following result, though rather simple, is very useful, for it allows us to replace a non-positive measure $\mu$ with a positive measure $\nu$ without losing any of the $\mu$-Martin-Löf random reals.

**Lemma 2.10** *If $\mu$ is a computable measure, and if $x \in \mathsf{MLR}_\mu$, then $x \in \mathsf{MLR}_{\frac{\mu+\lambda}{2}}$.*

**PROOF.** Suppose $x \notin \mathsf{MLR}_{\frac{\mu+\lambda}{2}}$. Then if $\{\mathcal{U}_i\}_{i\in\omega}$ is a $\frac{\mu+\lambda}{2}$-Martin-Löf test such that $x \in \bigcap_{i\in\omega}\mathcal{U}_i$, it follows that

$$\frac{\mu(\mathcal{U}_i) + \lambda(\mathcal{U}_i)}{2} \leq 2^{-i}$$

and hence

$$\mu(\mathcal{U}_i) \leq \mu(\mathcal{U}_i) + \lambda(\mathcal{U}_i) \leq 2^{-i+1}.$$

Thus $\{\mathcal{U}_{i+1}\}_{i\in\omega}$ is a $\mu$-Martin-Löf test containing $x$, and hence $x \notin \mathsf{MLR}_\mu$.

**Definition 2.11** *Given a computable measure $\mu$, a* $\mu$-Schnorr test *is a Martin-Löf test $\{\mathcal{U}_i\}_{i\in\omega}$ with the additional requirement that $\mu(\mathcal{U}_i)$ is a computable real number, uniformly in $i$. Furthermore, a real $x$ is* $\mu$-Schnorr random *if for every $\mu$-Schnorr test $\{\mathcal{U}_i\}_{i\in\omega}$, we have $x \notin \bigcap_{i\in\omega}\mathcal{U}_i$. The collection of $\mu$-Schnorr random reals will be written as $\mathsf{SR}_\mu$.*

**Definition 2.12** *A* computable $\mu$-martingale *is a computable function $d : 2^{<\omega} \to \mathbb{R}^{\geq 0} \cup \{+\infty\}$ such that for every $\sigma \in 2^{<\omega}$,*

$$\mu(\sigma)d(\sigma) = \mu(\sigma 0)d(\sigma 0) + \mu(\sigma 1)d(\sigma 1)$$

*(with the convention $0 \cdot (+\infty) = 0$). A computable $\mu$-martingale $d$ succeeds on $x \in 2^\omega$ if*

$$\limsup_{n\to\infty} d(x{\upharpoonright}n) = +\infty.$$

**Definition 2.13** *A real $x \in 2^\omega$ is* $\mu$-computably random *if there is no computable $\mu$-martingale $d$ that succeeds on $x$. The collection of $\mu$-computably random reals will be written as $\mathsf{CR}_\mu$.*

When we speak about a Martin-Löf random real (resp. computably random, Schnorr random) without specifying the measure, we mean "random with respect to the Lebesgue measure". Accordingly, we denote by $\mathsf{MLR}$ (resp. $\mathsf{CR}$, $\mathsf{SR}$) the set of Martin-Löf random (resp. computably random, Schnorr random) reals with respect to Lebesgue measure.

It is well known that $\mathsf{MLR}_\mu \subseteq \mathsf{CR}_\mu \subseteq \mathsf{SR}_\mu$ for every computable measure $\mu$ (the inclusions being strict for Lebesgue measure), a result that we will make use of in Section 4. Moreover, we will need two important results relating randomness notions and computational content. Recall that a real $x$ is high if $x' \geq_T \emptyset''$ or equivalently if there is an $x$-computable function $g : \mathbb{N} \to \mathbb{N}$ which dominates all computable functions (i.e., for every computable $f : \mathbb{N} \to \mathbb{N}$, for all but finitely many $n$, $f(n) \leq g(n)$).

**Theorem 2.14 ([NST05])** *For any computable measure $\mu$, if $x \in \mathsf{SR}_\mu \setminus \mathsf{MLR}_\mu$, then $x$ is high.*

We should note that this theorem was proven by Nies, Stephan, and Terwijn only in the case where $\mu$ is Lebesgue measure, but the entire argument trivially goes through when $\mu$ is taken to be an arbitrary computable measure.

**Theorem 2.15 ([NST05])** *For every high Turing degree* **a***, there is some $x \in \mathsf{CR}$ such that $x \in$* **a** *(moreover, $x$ can be taken outside $\mathsf{MLR}$).*

### *2.4 Almost-total functionals are defined on random reals*

By definition, the domain of a Turing reduction $\Phi$ is a $\Pi^0_2$ set of reals (indeed, it can be written as $\bigcap_n \mathcal{D}_n$ where $\mathcal{D}_n$ is the – effectively open – set of reals on which $\Phi$ produces at least $n$ bits of output). Therefore, given a (computable) measure $\mu$, a $\mu$-almost total Turing reduction is defined everywhere except on a $\Sigma^0_2$ set of $\mu$-measure 0 which in particular is a countable union of $\Pi^0_1$-classes of $\mu$-measure 0. The reals which do not belong to any $\Pi^0_1$ class of $\mu$-measure 0 are called *Kurtz random* or *weakly random*. It is well-known (see [DH10]) that Kurtz randomness is weaker than Schnorr randomness (and a fortiori computable randomness and Martin-Löf randomness). Thus we have the following important observation: a $\mu$-almost total functional $\Phi$ is defined on all $\mu$-Schnorr random (resp. $\mu$-computably random, $\mu$-Martin-Löf random) reals.

Moreover, if $\Phi$ is $\mu$-almost total, it is easy to construct a single $\mu$-Schnorr test $\{\mathcal{U}_i\}_{i\in\omega}$ such that $x \notin \mathrm{dom}(\Phi) \Rightarrow x \in \bigcap_{i\in\omega} \mathcal{U}_i$ (this follows from the proof that Schnorr randomness implies Kurtz randomness). This has the following consequence.

**Proposition 2.16** *Let $\mu$ be a computable measure and $\Phi$ a $\mu$-almost total Turing functional. Let $x$ be $\mu$-Schnorr random real. Then $y = \Phi(x)$ is defined, and $y$ is tt-reductible to $x$.*

**PROOF.** By the above discussion $y = \Phi(x)$ is well-defined. To see that it is *tt*-reducible to $x$, consider a $\mu$-Schnorr test $\{\mathcal{U}_i\}_{i\in\omega}$ such that $x \notin \mathrm{dom}(\Phi) \Rightarrow x \in \bigcap_{i\in\omega} \mathcal{U}_i$. Since $x$ is Schnorr random, it belongs to the complement $\mathcal{C}_k$ of $\mathcal{U}_k$ for some $k$, and by assumption $\Phi$ is defined on all elements of $\mathcal{C}_k$. It is a well-known fact that for every functional $\Phi$ which is defined on all elements of a $\Pi^0_1$ class $\mathcal{C}$, there exists a total functional $\Psi$ which coincides with $\Phi$ on $\mathcal{C}$ (it suffices, on an input $x$ to run $\Phi$ on $x$ while enumerating the complement $\mathcal{U}$ of $\Phi$ in parallel, and if at some stage $x$ is covered by $\mathcal{U}$, start outputting an infinite sequence of zeroes).

## 3 Demuth's Theorem

Informally, Demuth's Theorem tells us that when we apply an effectively continuous procedure to a Martin-Löf random real $x$, if the resulting real $y$ has *any* computational content whatsoever (i.e. is not computable), then from $y$ we can effectively recover a Martin-Löf random real $z$. Formally, the statement is as follows.

**Theorem 3.1 (Demuth [Dem88])** *Let $x$ be a Martin-Löf random real. Suppose $x$ computes a real $y$ via a $\lambda$-almost total reduction (or equivalently, computes $y$ via tt-reduction) and $y$ is not computable. Then $y$ is Turing equivalent to some Martin-Löf random real $z$.*

The best way to prove Demuth's Theorem is to break it down into two results, which have been shown by Levin, Kautz and Kurtz independently of Demuth's work. The first result is the well-known *Conservation of Randomness Theorem.* Not only do total and almost total functionals induce computable measures (as discussed in the previous section), but according to the Conservation of

Randomness Theorem, they also do so in such a way as to map reals random with respect to the original measure to reals random with respect to the induced measure; namely, we have the following result.

**Theorem 3.2 (Conservation of Martin-Löf randomness)** *Let $\mu$ be a computable measure and $\Phi$ a $\mu$-almost total functional. Then $x \in \mathsf{MLR}_\mu$ implies $\Phi(x) \in \mathsf{MLR}_{\mu_\Phi}$.*

**PROOF.** See Appendix.

The second result used to derive Demuth's Theorem is sometimes referred to as "Levin's Theorem" or "the Levin-Kautz Theorem" (although Levin proved the theorem with Zvonkin, and independently of Kautz).

**Theorem 3.3 (Levin/Zvonkin [LZ70], Kautz [Kau91])** *If $y$ is a real that is non-computable and $\mu$-Martin-Löf random for some computable measure $\mu$, then there is a Martin-Löf random real $z$ such that $y \equiv_T z$.*

Note that the two theorems given above immediately imply Demuth's Theorem: Given a *tt*-functional $\Phi$ and $x \in \mathsf{MLR}$, since $\Phi$ is almost total, by the Conservation of Randomness, it follows that $\Phi(x) \in \mathsf{MLR}_{\lambda_\Phi}$, and $\lambda_\Phi$ is computable by Lemma 2.6. Further, if $\Phi(x)$ is not computable, then by the Levin-Kautz Theorem there is some $z \in \mathsf{MLR}$ such that $\Phi(x) \equiv_T z$, thus establishing the result.

In order to prove the Levin-Kautz Theorem, we need to prove an auxiliary result that we will refer to as the *Kautz conversion procedure*. This result provides a converse to Lemmas 2.6 and 2.7, as it shows that any computable measure $\mu$ can be induced by the Lebesgue measure together with an almost total functional.

**Theorem 3.4 (The Kautz conversion procedure)** *Let $\mu$ be a computable probability measure. Then there exists a non-decreasing, almost total functional $\Phi$ such that $\lambda_\Phi = \mu$. Moreover,*

- *if $\mu$ is atomless, then $\Phi$ is one-to-one on its domain; and*
- *if $\mu$ is positive, then the range of $\Phi$ has $\mu$-measure 1.*

*Finally, if $\mu$ is both atomless and positive, then $\Phi$ has an almost total inverse $\Phi^{-1}$ such that $\mu_{\Phi^{-1}} = \lambda$.*

**PROOF.** See Appendix.

The next theorem, first proven by Shen (unpubl.), can be seen as a partial converse to the Conservation of Randomness Theorem, stating that every sequence that is random with respect to a computable measure induced by a functional $\Phi$ has a random real in its preimage under $\Phi$.

**Theorem 3.5** *Let $\mu$ be a computable measure, $\Phi$ a $\mu$-almost total functional, and $y \in 2^\omega$. If $y \in \mathsf{MLR}_{\mu_\Phi}$, then there is some $x \in \mathsf{MLR}_\mu$ such that $\Phi(x) = y$.*

**PROOF.** Suppose $y \in 2^\omega$ is such that for all $x \in 2^\omega$, $x \in \mathsf{MLR}_\mu$ implies $\Phi(x) \neq y$. Then if $\{\mathcal{U}_i\}_{i\in\omega}$ is the universal $\mu$-Martin-Löf test, consider

$$\mathcal{V}_i = \{z \in 2^\omega : (\forall x)[x \notin \mathcal{U}_i \Rightarrow \Phi(x) \neq z]\}.$$

We claim that $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mu_\Phi$-Martin-Löf test. First, observe that $z \in \mathcal{V}_i$ if and only if $z \notin \Phi(2^\omega \setminus \mathcal{U}_i)$. Since $\Phi$ is an almost total Turing functional, the image under $\Phi$ of a $\Pi^0_1$ class is also a $\Pi^0_1$ class. In particular, $\Phi(2^\omega \setminus \mathcal{U}_i)$ is a $\Pi^0_1$ class, and so $\mathcal{V}_i$ is a $\Sigma^0_1$ class. Further, $\{\mathcal{V}_i\}_{i\in\omega}$ is clearly uniformly $\Sigma^0_1$.

To see that $\mu_\Phi(\mathcal{V}_i) \leq 2^{-i}$, since $2^\omega \setminus \mathcal{U}_i \subseteq \Phi^{-1}(\Phi(2^\omega \setminus \mathcal{U}_i))$, we have

$$\begin{aligned}\mu_\Phi(\mathcal{V}_i) = 1 - \mu_\Phi(\Phi(2^\omega \setminus \mathcal{U}_i)) &= 1 - \mu(\Phi^{-1}(\Phi(2^\omega \setminus \mathcal{U}_i))) \\ &\leq 1 - \mu(2^\omega \setminus \mathcal{U}_i) \\ &\leq 1 - (1 - 2^{-i}) = 2^{-i}.\end{aligned}$$

Lastly, since for all $x \in 2^\omega$ such that $\Phi(x) = y$, $x \notin \mathsf{MLR}_\mu$, it follows that $x \notin \mathcal{U}_i$ implies $\Phi(x) \neq y$ for every $i$, and so $y \in \mathcal{V}_i$ for every $i$. Thus $y \notin \mathsf{MLR}_{\mu_\Phi}$.

We should note that if we were to define the above $\mu_\Phi$-Martin-Löf test $\{\mathcal{V}_i\}_{i\in\omega}$ in terms of a Martin-Löf test $\{\mathcal{U}_i\}_{i\in\omega}$ that is not universal, then it wouldn't necessarily follow that $y \in \mathcal{V}_i$ for every $i$, since there may be some non-random $z \notin \bigcap_{i\in\omega} \mathcal{U}_i$ such that $\Phi(z) = y$. For this reason, the above proof does not work if we consider, instead of Martin-Löf randomness, a notion of randomness for which there is no universal test, such as Schnorr randomness.

We can now prove the Levin-Kautz Theorem.

**PROOF of Theorem 3.3** Given a non-computable $y$ and a computable measure $\mu$ such that $y \in \mathsf{MLR}_\mu$, by Theorem 3.4, there is some non-decreasing $\lambda$-almost total functional $\Phi$ such that $\mu = \lambda_\Phi$. Since $y \in \mathsf{MLR}_{\lambda_\Phi}$, by Theorem 3.5, there is some $z \in \mathsf{MLR}$ such that $\Phi(z) = y$. Moreover, suppose there were another real $u$ in the preimage of $y$ under $\Phi$. Since $\Phi$ is non-decreasing, this would mean that the whole interval $[z, u]$ (or $[u, z]$) is entirely mapped by $\Phi$ to the singleton $\{y\}$, and therefore $y$ would be an atom of $\lambda_\Phi$, hence would be computable, a contradiction. Therefore, $y$ has a unique preimage $z$ under $\Phi$; in other words $\Phi^{-1}(\{y\})$ is a $\Pi^0_1(y)$-class containing a single element $z$, hence $z$ is $y$-computable. We have proven $z \equiv_T y$ and $z \in \mathsf{MLR}$, as wanted.

## 4 Demuth's Theorem for other notions of randomness

In this section, we consider Demuth's Theorem for other notions of effective randomness, namely Schnorr randomness and computable randomness. We will show that is holds for both notions. The first step towards this result is the following analogue of Theorem 3.2 (Conservation of Randomness) for Schnorr randomness.

**Theorem 4.1 (Conservation of Schnorr randomness)** *Let $\mu$ be a computable measure and $\Phi$ an almost total functional. Then $x \in \mathsf{SR}_\mu$ implies $\Phi(x) \in \mathsf{SR}_{\mu_\Phi}$.*

**PROOF.** In the proof of the conservation of Martin-Löf randomness (Theorem 3.2) in the Appendix, we show that if $\Phi(x)$ is contained in a $\mu_\Phi$-Martin-Löf test $\{\mathcal{U}_i\}_{i\in\omega}$, then there is a $\mu$-Martin-Löf test $\{\mathcal{V}_i\}_{i\in\omega}$ containing $x$. But in fact, we prove more: we show that $\mu(\mathcal{V}_i) = \mu_\Phi(\mathcal{U}_i)$. Thus, if $\{\mathcal{U}_i\}_{i\in\omega}$ is a $\mu_\Phi$-Schnorr test containing $\Phi(x)$, it follows that $\{\mathcal{V}\}_{i\in\omega}$ is a $\mu$-Schnorr test containing $x$.

Perhaps surprisingly, there is no Conservation of Randomness Theorem for computable randomness (a result independently proven by Rute [?]).

**Theorem 4.2** *There exists a tt-reduction $\Phi$ that does not preserve computable randomness, that is, for some computably random $x$, $\Phi(x)$ is not computably random for the measure induced by $\Phi$. One can even construct an example where $\Phi$ induces the Lebesgue measure.*

**PROOF.** This result follows from the work of Muchnik who proved that Kolmogorov-Loveland randomness is stronger than computable randomness. A Kolmogorov-Loveland random real is a real that defeats all computable non-monotonic strategies, where a non-monotonic strategy is a betting strategy which at each turn chooses which bit of the sequence (that has not been revealed so far) it will bet on, and then bets on the value of the bit. It was proven in [MSU98] that Kolmogorov-Loveland randomness is strictly stronger than computable randomness, and in [MMN$^+$06] that in the definition of Kolmogorov-Loveland randomness, one can assume that only *total* non-monotonic strategies are allowed. Consider therefore a sequence $x \in 2^\omega$ which is computably random but not Kolmogorov-Loveland random. Let $S$ be a total non-monotonic strategy that defeats $x$. Now, define $\Phi$ to be the *tt*-functional which to a sequence $z$ associates the sequence $y$ of the bits of $z$ seen by $S$ during the game (in order of appearance). Certainly $\Phi$ is a *tt*-reduction and induces the Lebesgue measure. By definition of Kolmogorov-Loveland randomness, $\Phi(x)$ is not computably random, which proves the result. We can also invoke the stronger result by Kastermans and Lempp [KL10] who proved that computable randomness is not closed under computable injective re-orderings of bits (i.e. there exists a computably random real $x$ and a computable injective function $f$ such that $x(f(0))x(f(1))\dots$ is not computably random).

The proof of the Levin-Kautz Theorem that we provided in the previous section does not work for Schnorr randomness, since the proof relies upon the existence of a universal Martin-Löf test (via Theorem 3.5), and it is well-known that there is no universal Schnorr test (see the explanation given after the proof of Theorem 3.5). For computable randomness, the situation is even worse, as we have seen that there is no analogue of the Conservation of Randomness Theorem for this notion. As a consequence, we cannot prove Demuth's Theorem for these two randomness notions by a direct adaptation of the proof for Martin-Löf randomness. However, there is an interesting way to overcome the difficulty, which works both for computable randomness and Schnorr randomness. This alternative approach uses the results of Nies, Stephan, and Terwijn mentioned in Section 2 (see Theorem 2.14): a Schnorr random (resp. computably random) real is either Martin-Löf random, or it is high. Armed with this dichotomy, we get Demuth's Theorem for Schnorr randomness and computable randomness almost immediately from Demuth's Theorem for Martin-Löf randomness. In fact, we get a slightly stronger statement that subsumes both, in the sense that it suffices to assume $x \in \mathsf{SR}$ to get $z \in \mathsf{CR}$ in the conclusion.

**Theorem 4.3 (Demuth's Theorem for $\mathsf{CR}$ and $\mathsf{SR}$)** *Let $x \in \mathsf{SR}$ and let $\Phi$ be a truth-table functional. If $\Phi(x) = y$ is not computable, then there is some $z \in \mathsf{CR}$ such that $y \equiv_T z$.*

**PROOF.** Whether $x \in \mathsf{CR}$ or $x \in \mathsf{SR}$ (the latter being the weaker assumption), the Conservation of Schnorr randomness (Theorem 4.1) implies that $y$ is Schnorr random with respect to some computable measure $\mu$. We now distinguish two cases.

Case 1. If $y$ is not high, then by Theorem 2.14 it must be $\mu$-Martin-Löf random. Thus, we can apply the Levin-Kautz Theorem (Theorem 3.3) to get a real $z \equiv_T y$ that is Martin-Löf random (hence computably random).

Case 2. If $y$ is high, then we can directly apply Theorem 2.15 to get the existence of some $z \in \mathsf{CR}$ such that $z \equiv_T y$.

## 5 The failure of Demuth's Theorem for *wtt*-reducibility

In the original proof of Demuth's Theorem, Demuth shows that in the conclusion of the theorem, one can even require $y \equiv_T z$ *and* $y \leq_{tt} z$. Hence we actually have a stronger version of Theorem 3.1.

**Theorem 5.1** *Let $x$ be a Martin-Löf random real. Suppose $x$ tt-computes a non-computable real $y$. Then $y$ is Turing equivalent to some Martin-Löf random real $z$, and furthermore $y \leq_{tt} z$.*

It is therefore natural to ask whether the reverse reduction, $y \geq_T z$, can also be required to be stronger (*tt* or *wtt*). We will prove that this is not the case in general, not only for Demuth's original theorem, but also for the versions of Demuth's Theorem for computable randomness and Schnorr randomness proven in the previous section.

Our analysis will make use of the so-called complex reals, which were defined by Kjos-Hanssen et al. [KHMS11]. To prove the failure of the *wtt*-version of Demuth's Theorem, we will show that (1) if a real $y$ *wtt*-computes a Martin-Löf random real $z$, it must be complex and (2) there is an $x \in \mathsf{MLR}$ and a real $y \leq_{tt} x$ such that $y$ is non-computable but not complex.

Complex reals were defined by Kjos-Hanssen et al. using plain (or prefix-free) Kolmogorov complexity. We shall define them using another version of Kolmogorov complexity, called *monotone complexity*, which for our purposes is slightly easier to handle. The fact that our definition is equivalent to theirs is proven in Proposition A.1 of the Appendix.

**Definition 5.2** *A monotone machine is a computable function $M : 2^{<\omega} \to 2^{<\omega} \cup 2^{\omega}$ such that $M(\sigma_1) \preceq M(\sigma_2)$ for all $\sigma_1 \preceq \sigma_2$ and the set of pairs of strings $(\sigma, \tau)$ with $\tau \preceq M(\sigma)$ is c.e. Fixing a universal monotone machine $M$, we define the $Km$-complexity of $\tau \in 2^{<\omega}$ to be*

$$Km(\tau) = \min\{|\sigma| : \tau \preceq M(\sigma)\}.$$

*A real $x$ is said to be complex if there is a computable, non-decreasing, unbounded function $g$ such that $Km(x{\upharpoonright}n) \geq g(n)$ for every $n$.*

In the sequel we will refer to a non-decreasing, unbounded function from $\omega$ to $\omega$ as an *order function*. For any order function $g$, $g^{-1}$ is the order function defined by

$$g^{-1}(n) = \min\{k : g(k) \geq n\}$$

The Levin-Schnorr Theorem states that a real $z$ is Martin-Löf random (with respect to the Lebesgue measure) if and only if $Km(z\restriction n) = n - O(1)$; in particular, Martin-Löf random reals are complex. Furthermore, any real $y$ that $wtt$-computes a Martin-Löf random real is itself complex. This follows from the straightforward fact that complex reals are closed upwards in the $wtt$-degrees.

**Lemma 5.3** *Let $a, b$ be two reals such that $a \geq_{wtt} b$. If $b$ is complex then so is $a$.*

**PROOF.** Indeed let $\varphi$ be a computable bound for the use of the $wtt$-reduction. Suppose $b$ is complex with order function $g$. Then

$$g(\varphi^{-1}(n)) \leq Km(b\restriction\varphi^{-1}(n)) \leq Km(a\restriction n)$$

therefore $a$ is complex via $g \circ \varphi^{-1}$ which is a computable order function.

The second step of the proof requires more effort.

**Theorem 5.4** *There exists $a \in \mathsf{MLR}$ and a non-computable real $b \leq_{tt} a$ which is not complex.*

**PROOF.** To prove this theorem, we take $a$ to be Chaitin's $\Omega$ number, where $\Omega := \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$, $U$ being a universal prefix-free machine. It is well-known that $\Omega \in \mathsf{MLR}$ and is a left-c.e. real, which means that there is a computable sequence of rationals $(\Omega_s)_s$ that converges to $\Omega$ from below. Note that this sequence must converge very slowly, i.e. there is no computable function $f$ such that $\Omega\restriction n = \Omega_{f(n)}\restriction n$ infinitely often, for otherwise we would be able to compress the corresponding initial segments of $\Omega$. We use the slowness of this approximation to build our sparse real $b$. We achieve this through the following $tt$-reduction. Let $\Phi : 2^\omega \to 2^\omega$ be the functional (which we will call the "slowdown functional") defined for all reals by

$$\Phi(x) = 1^{t_1}01^{t_2}01^{t_3}0\ldots$$

where the $t_i$ are defined as follows: $t_0 = 0$ and

$$t_i = \min\{s : \Omega_s \geq 0.x\restriction i\}$$

with the convention that if the set on the right-hand side is empty, then $t_i = +\infty$. Thus if some $t_i$ is infinite, then $\Phi(x) = 1^{t_1}0 \dots 1^{t_k}011111\dots$ where $t_{k+1}$ is the first $t_i$ to be infinite.

$\Phi$ is clearly a $tt$-reduction. Moreover, if $a < \Omega$, then there is some $s$ such that $\Omega_s > a\restriction i$ for every $i \in \omega$, and hence

$$\Phi(a) = \sigma(1^k0)^\omega$$

for some $\sigma \in 2^{<\omega}$ and $k \in \omega$. If $a > \Omega$, then there is some $i$ such that $\Omega_s < a\restriction i$ for every $s \in \omega$, and hence

$$\Phi(a) = \sigma 1^\omega$$

for some $\sigma \in 2^{<\omega}$.

The interesting case is when $a = \Omega$, for in this case, setting

$$\Phi(\Omega) = 1^{s_1}01^{s_2}01^{s_3}0\dots,$$

we know that the function $f$ given by $f(i) = s_i$ grows faster than any computable function, since $\Omega\restriction n = \Omega_{f(n)}\restriction n$ for every $n \in \omega$. If we set $\Phi(\Omega) = \Omega^*$, then we have

$$Km(\Omega^*\restriction f(n)) \leq^+ n$$

and hence

$$Km(\Omega^*\restriction n) \leq^+ f^{-1}(n),$$

But since $f$ grows faster than any computable order function, $f^{-1}$ is dominated by all computable order functions. Thus, there is no computable order function $g$ such that

$$Km(\Omega^*\restriction n) \geq g(n).$$

We can now prove that the $wtt$-version of Demuth's theorem fails for Martin-Löf randomness.

**Corollary 5.5** *There exists $a \in \mathsf{MLR}$ and a non-computable real $b \leq_{tt} a$ such that there is no $y \in \mathsf{MLR}$ with $y \leq_{wtt} b$.*

**PROOF.** By Theorem 5.4, $\Omega^*$ is $tt$-reducible to a Martin-Löf random real but is not complex. Hence by Lemma 5.3, $\Omega^*$ cannot $wtt$-compute any complex real, and thus $\Omega^*$ cannot $wtt$-compute any $x \in \mathsf{MLR}$.

There are only countably many reals that are random with respect to the measure induced by the slowdown functional $\Phi$ in the proof of Theorem 5.4 (and all of them are atoms except for $\Omega^*$), but this does not have to be the case, as is shown by the following result.

**Proposition 5.6** *There is a computable measure $\mu$ such that there are continuum many $x \in \mathsf{MLR}_\mu$ and continuum many such $x$ that do not wtt-compute any $y \in \mathsf{MLR}$.*

**PROOF.** We define a new functional $\Psi$ that on input $a \oplus b$ behaves similarly to the slowdown functional $\Phi$ defined in the proof of Theorem 5.4. Suppose that $\Phi(a) = 1^{t_1}01^{t_2}01^{t_3}0 \dots 1^{t_i}0 \dots$. Then we have

$$\Psi(a \oplus b) = b_0^{t_1} b_1^{t_2} b_2^{t_3} \dots b_i^{t_i} \dots$$

where $b_i = b(i)$ for every $i$. Note that $\Psi$ is total, since $\Phi$ is total. Further, if $B$ is 2-random (i.e. $b \in \mathsf{MLR}^{\emptyset'}$), then $b \in \mathsf{MLR}^{\Omega}$ and hence $\Omega \oplus b \in \mathsf{MLR}$ by van Lambalgen's Theorem (according to which $A \oplus B \in \mathsf{MLR} \Leftrightarrow A \in \mathsf{MLR}^B$ and $B \in \mathsf{MLR}$ for any $A, B \in 2^\omega$; see [DH10], Chapter 6.9). It follows from the Conservation of Martin-Löf randomness that $\Psi(\Omega \oplus b)$ is random with respect to the induced measure $\lambda_\Psi$. Moreover, as with $\Omega^*$, $\Psi(\Omega \oplus b)$ is not complex, and thus cannot *wtt*-compute any $y \in \mathsf{MLR}$.

As we have seen, there are many counterexamples to *wtt*-generalization of Demuth's Theorem for Martin-Löf randomness. We now show that the *wtt*-generalizations of Demuth's Theorem for computable randomness and Schnorr randomness also fail to hold. It seems that the real $\Omega^*$ constructed in the proof of Theorem 5.4 is so far from complex that it should not even *wtt*-compute a Schnorr random real. Unfortunately, we do not know whether this is the case. We therefore need to slightly adapt the technique used in the proof of Theorem 5.4, still keeping the main ideas.

**Theorem 5.7** *For almost all reals $a$, there exists a non-computable real $b \leq_{tt} a$ which does not wtt-compute any Schnorr random real.*

This shows in particular that there exists a Martin-Löf random (hence computably random Schnorr random) real $a$, and a non-computable $b \leq_{tt} a$ which does not *wtt*-compute any Schnorr random real. Therefore the *wtt*-version of Demuth's Theorem fails for all three notions of randomness.

To prove Theorem 5.7, we need a few auxiliary facts.

**Lemma 5.8** *Let $f$ be an increasing function that is not dominated by any computable function. Let $g$ be a computable order function. Then for infinitely many $n$,*

$$f(n) < g(f(n+1)).$$

**PROOF.** Indeed, if the opposite holds, i.e., $f(n+1) \leq m(f(n))$ for all $n \geq k$, where $m$ is the inverse function of $g$, then it is easy to show by induction that

$$f(n) \leq m^{(n-k)}(f(k))$$

for all $n \geq k$. The right-hand side of the above expression being a nondecreasing and computable function of $n$, we have a contradiction.

**Proposition 5.9** *Let $a$ be a Martin-Löf random real of hyperimmune degree. Then there is a real $b \leq_{tt} a$ such that $b$ is not complex.*

**PROOF.** Since $a$ is of hyperimmune degree, it computes a function $f$ which is infinitely often above any given computable function $g$. Let $\psi$ be a partial computable function such that, when equipped with the real $a$ as an oracle, computes the function $f$. For all $n$, define

$$g(n) = \min\{t : \psi^a[t](n)\downarrow\}$$

By the standard conventions on oracle computations, it follows that $g(n) \geq f(n)$ for all $n$ (as we require that the number of steps for a halting computation always exceeds the output of the computation). It follows that $g$ is not dominated by any computable function. Now let $\Theta$ be the reduction defined by

$$\Theta(x) = 1^{t_0}01^{t_1}01^{t_2}\ldots$$

with

$$t_n = \min\{t : \psi^x[t](n)\downarrow\}$$

(with the convention that $\Theta(x) = 1^{t_0}01^{t_1}01\ldots1^{t_{i-1}}01111111\ldots$ if $t_i$ is infinite and is the smallest such $t_n$). The definition ensures that $\Theta$ is total and that

$$b = \Theta(a) = 1^{g(0)}01^{g(1)}01^{g(2)}\ldots$$

We need to show that $b$ is not complex. Let $h$ be a computable order function. Notice that

$$\begin{aligned} Km(1^{g(0)}01^{g(1)}\ldots01^{g(n)}01^{g(n+1)}) &\leq K(1^{g(0)}01^{g(1)}\ldots01^{g(n)}0) + O(1) \\ &\leq n \cdot \log g(n) + O(1) \\ &\leq g(n)\log g(n) + O(1), \end{aligned}$$

where the first inequality follows from two facts: (i) $Km(\sigma\tau) \leq K(\sigma) + Km(\tau) + O(1)$ and (ii) $Km(1^k) = O(1)$ for all $k$. By Lemma 5.8 applied to the composition of $h$ and $(n \mapsto n\log n)^{-1}$, we have for infinitely many $n$, $g(n)\log g(n) + k < h(g(n+1))$ for any fixed $k \in \omega$ (as $g(n)\log g(n) + k$ is not dominated by any computable function). Thus for infinitely many $n$,

$$Km(b{\upharpoonright}g(n+1)) \leq Km(1^{g(0)}01^{g(1)}\ldots01^{g(n)}01^{g(n+1)}) < h(g(n+1)).$$

Since this is the case for any order function $h$, it follows that $b$ is not complex.

We are now ready to prove Theorem 5.7.

**PROOF of Theorem 5.7** Let $a$ be a random real of hyperimmune but non-high degree. Note that almost all reals have this property. More precisely, any 3-random[1] is a real has this property: any 2-random real has hyperimmune degree, as proven by Kurtz [Kur81] and no 3-random real is high [Nie09, Exercise 8.5.21]. Then by Proposition 5.9, $a$ *tt*-computes a real $b$ which is not complex. Now suppose $b$ *wtt*-computes a real $c$. Then since $b$ is not complex, by Lemma 5.3, $c$ is not complex. In particular, $c$ is not Martin-Löf random (recall that a Martin-Löf random real $z$ is s.t. $Km(z\restriction n) = n - O(1)$). Moreover, $c$ is not high, as $a \geq_T b \geq_T c$ and $a$ is not high. Therefore by Theorem 2.14, if $c$ is not Martin-Löf random and not high, then $c$ cannot be Schnorr random.

## 6 On the degrees of random reals

### *6.1 Random Turing degrees*

One consequence of the machinery developed in the previous section is that we can use it to provide an exact characterization of all the Martin-Löf random Turing degrees that contain a real that is random with respect to a computable measure but not random with respect to any computable *atomless* measure (recall that a Turing degree is Martin-Löf random if it contains a Martin-Löf random real). Let us establish a few more definitions that will be useful in this section.

**Definition 6.1** *Let* $\mathsf{MLR}_{\text{comp}}$ *be the set of reals* $x$ *such that* $x \in \mathsf{MLR}_\mu$ *for some computable measure* $\mu$.

The class $\mathsf{MLR}_{\text{comp}}$ was first considered in [LZ70], where elements of $\mathsf{MLR}_{\text{comp}}$ were referred to as "proper sequences", and was later studied further in [SF77] and [MSU98].

**Definition 6.2** *Let* $\mathsf{NCR}_{\text{comp}}$ *be the set of reals* $x$ *such that* $x \notin \mathsf{MLR}_\mu$ *for every computable atomless measure* $\mu$.

[1] For $n \geq 1$, a real $x$ is $n$-random if and only if $x \in \mathsf{MLR}^{\emptyset^{(n-1)}}$, that is, $x$ is Martin-Löf random relative to $\emptyset^{(n-1)}$.

The motivation behind the definition of $\mathsf{NCR}_{\text{comp}}$ comes from the work of Reimann and Slaman (see, for instance, [RS07] and [RS08]), who studied the collection of sequences that are not random with respect to *any* atomless measure (computable or otherwise), referring to this class as $\mathsf{NCR}_1$. Although Reimann and Slaman have established a number of facts about $\mathsf{NCR}_1$, for instance, that it is countable and contains no non-$\Delta^1_1$ reals, a number of questions about the structure of $\mathsf{NCR}_1$ remain open. $\mathsf{NCR}_{\text{comp}}$, in contrast, proves to be much easier to characterize.

We will begin by showing that there is at least one $x \in \mathsf{MLR}_{\text{comp}} \cap \mathsf{NCR}_{\text{comp}}$.

**Proposition 6.3** *There is a non-computable $x \in 2^\omega$ that is random with respect to some computable atomic measure but not random with respect to any computable atomless measure.*

To prove this proposition, we need one further result. In Section 2 we saw that if a computable measure $\mu$ is atomless and positive, then if $\Phi$ is an almost total functional such that $\lambda_\Phi = \mu$, then $\Phi^{-1}$ is an almost total functional such that $\mu_{\Phi^{-1}} = \lambda$. However, if $\Phi$ is total, it doesn't necessary follow that $\Phi^{-1}$ is total, but we can still obtain a measure $\nu$ induced by some *other* *tt*-functional such that $\nu$ is equivalent to $\lambda$, in the sense that $\mathsf{MLR}_\nu = \mathsf{MLR}$.

**Proposition 6.4** *If $\mu$ is an atomless, computable measure, then there is a non-decreasing tt-functional $\Theta$ such that the induced measure $\mu_\Theta$ has the property that*

$$\mathsf{MLR}_{\mu_\Theta} = \mathsf{MLR}.$$

**PROOF.** See Appendix.

**PROOF of Proposition 6.3.** The real constructed in the proof of Theorem 5.4 above, $\Omega^*$, is random with respect to the induced measure $\lambda_\Phi$ (which is clearly atomic), and hence $\Omega^* \in \mathsf{MLR}_{\text{comp}}$. Suppose, for sake of contradiction, that $\Omega$ is random with respect to a computable, atomless measure $\mu$. Then by Proposition 6.4, there is a *tt*-functional $\Theta$ such that $\mathsf{MLR}_{\mu_\Theta} = \mathsf{MLR}$. Moreover, by the Conservation of Randomness Theorem, it follows that $\Theta(\Omega^*) \in \mathsf{MLR}_{\mu_\Theta} = \mathsf{MLR}$, but as we proved in Theorem 5.4, $\Omega^*$ can't even *wtt*-compute any $y \in \mathsf{MLR}$, yielding the desired contradiction. Thus $\Omega^* \in \mathsf{NCR}_{\text{comp}}$.

We can use the idea of this proof to provide a full classification of the Martin-Löf random Turing degrees containing elements in $\mathsf{MLR}_{\text{comp}} \cap \mathsf{NCR}_{\text{comp}}$. In providing the classification, we will use the following.

**Proposition 6.5 ([RS08], Proposition 5.7)** *For $a \in \mathsf{MLR}$ and $b \in 2^\omega$, if $a \equiv_{tt} b$, then $b \notin \mathsf{NCR}_{\mathrm{comp}}$.*

**Theorem 6.6** *Let $\mathbf{a}$ be a Martin-Löf random Turing degree. Then there is some $a \in \mathbf{a}$ such that $a \in \mathsf{MLR}_{\mathrm{comp}} \cap \mathsf{NCR}_{\mathrm{comp}}$ if and only if $\mathbf{a}$ is hyperimmune.*

**PROOF.** For the easier direction, suppose $\mathbf{a}$ is hyperimmune-free. Then given $a \in \mathbf{a} \cap \mathsf{MLR}$, by a well-known result, if $b \equiv_T a$, then $b \equiv_{tt} a$. Thus for any $b \equiv_T a$, by the Conservation of Randomness Theorem we have $b \in \mathsf{MLR}_{\mathrm{comp}}$, but by Proposition 6.5, $b \equiv_{tt} a$ implies that $b \notin \mathsf{NCR}_{\mathrm{comp}}$, i.e. $b$ is random with respect to some atomless measure. Thus no $b \in \mathbf{a}$ is in $\mathsf{MLR}_{\mathrm{comp}} \cap \mathsf{NCR}_{\mathrm{comp}}$.

Now suppose that $\mathbf{a}$ is hyperimmune, and let $a \in \mathbf{a} \cap \mathsf{MLR}$. We proceed as in the proof of Proposition 5.9, with a slight modification. Let $f \in \mathbf{a}$ be a function that is not dominated by any computable function. Then there is some partial computable function $\psi$ equipped with oracle $a$ such that $\psi^a(n) = f(n)$ for every $n$. Then, similar to the proof of Proposition 5.9, we define a functional $\Gamma$ such that

$$\Gamma(c) = 1^{t_0}\, 0^{c(0)+1}\, 1^{t_1}\, 0^{c(1)+1}\, 1^{t_2}\, 0^{c(2)+1} \dots,$$

where $t_i$ is the least $t$ such that $\Psi^c(i)[t]\downarrow$, unless no such $t$ exists, in which case $t_i = +\infty$. Note that we code the real $c$ into $\Gamma(c)$ so that if the $(i+1)$st block of 0s in $\Gamma(c)$ has length 1, then $c(i) = 0$, and if the $(i+1)$st block of 0s in $\Gamma(c)$ has length 2, then $c(i) = 1$. Thus we have $\Gamma(a) \equiv_T a$. Further, by the Conservation of Randomness Theorem, we have $\Gamma(a) \in \mathsf{MLR}_{\mathrm{comp}}$. Now let $g : \omega \to \omega$ be the function such that

$$\Gamma(a) = 1^{g(0)}\, 0^{a(0)+1}\, 1^{g(1)}\, 0^{a(1)+1}\, 1^{g(2)}\, 0^{a(2)+1} \dots$$

Given the convention that for the least $t$ such that $\Psi^c(i)[t]\downarrow = k$, we have $k \leq t$, it follows that $f(n) \leq g(n)$, and hence $g(n)$ is not dominated by any computable function.

Now, we verify $\Gamma(a)$ is not complex as before, with the only difference being that we now have to consider the potentially doubled 0s, yielding

$$Km(1^{g(0)}\, 0^{a(0)+1}\, 1^{g(1)}\, 0^{a(1)+1} \dots 1^{g(n)}\, 0^{a(n)+1} 1^{g(n+1)}) \leq 2n{\cdot}\log g(n) \leq g(n) \log g(n).$$

All the other steps proceed as before, and thus $\Gamma(a)$ is not complex. Now, assuming that $\Gamma(a)$ is random with respect to some atomless measure, we can argue as in the proof of Proposition 6.3 that $\Gamma(a)$ must $tt$-compute a Martin-Löf random real, contradicting the fact that $\Gamma(a)$ is not complex. Thus $\Gamma(a) \in \mathsf{NCR}_{\mathrm{comp}}$.

Recall that a real $x$ is weakly 1-generic if it is contained in every dense effectively open subset of $2^\omega$. Since every hyperimmune degree contains a weakly

1-generic real (as shown in [Kur81]) and no weakly 1-generic real is Martin-Löf random with respect to *any* computable measure (as is shown in [MSU98], Theorem 9.10), we have an even stronger dichotomy: Every hyperimmune-free random degree contains only reals that are random with respect to some computable atomless measure, while every hyperimmune random degree contains reals that are random only with respect to some computable atomic measure as well as reals that aren't random with respect to *any* computable measure.

### *6.2 Random computably enumerable sets*

In this last subsection, we will show that the Conservation of Randomness Theorem and related results also have consequences for the study of random computably enumerable sets. In particular, we show the existence of a computably enumerable set that is random with respect to some computable measure. This is somewhat surprising, given that computably enumerable sets quite far from Martin-Löf random. For instance, every c.e. set $x$ has low initial segment complexity: for every $n$, $K(x\restriction n) \leq 2\log(n) + O(1)$. Despite this behavior, there are c.e. sets that are Martin-Löf random with respect to *some* computable measure, as we now demonstrate (this result was obtained independently by Reimann and Slaman).

**Theorem 6.7** *There exists a non-computable c.e. set $x$ and a computable probability measure $\mu$ such that $A$ is random with respect to $\mu$.*

**PROOF.** Let $(q_n)_{n\in\omega}$ be an effective enumeration of $\mathbb{Q}_2$. Let $T : 2^\omega \to 2^\omega$ be the map defined by

$$T(x) = \{n \mid q_n < x\}$$

where we see the input as an infinite binary sequence and the output as a set of integers. Clearly $T$ is a computable one-to-one map, hence the measure $\mu$ it induces on $2^\omega$ is computable and atomless, and for every random $x$, $T(x)$ is $\mu$-random. If $x$ is left-c.e. by definition of $T$, $T(x)$ is a c.e. set. Therefore, $T(\Omega)$ is both c.e. and $\mu$-random.

We can also show that there is a non-computable c.e. member of $\mathsf{MLR}_{\mathrm{comp}} \cap \mathsf{NCR}_{\mathrm{comp}}$.

**Theorem 6.8** *There is a non-computable c.e. set $c$ such that $c \in \mathsf{MLR}_\mu$ for some computable atomic measure $\mu$ but $c \notin \mathsf{MLR}_\nu$ for any computable atomless measure $\nu$.*

**PROOF.** To prove this result, we merely need to show that $\Omega^*$, the real constructed in the proof of Theorem 5.4, is left-c.e. and then apply the map $T$ defined above to produce a c.e. set $C$ that is $tt$-reducible to $\Omega$ (via the composition of $T$ with the slowdown operator $\Phi$ defined in the proof of Theorem 5.4). It will then follow that $C$ cannot be random with respect to any atomless computable measure, for as we argued in the proof of Proposition 6.3, this would mean that $C$, and hence $\Omega^*$, can $tt$-compute a 1-random.

To see that $\Omega^*$ is left-c.e., notice that $\Phi$ is a non-decreasing functional and $\Phi$ is continuous at $\Omega$. Therefore, for a rational $q$, we have

$$q < \Omega^* \Leftrightarrow \exists x \; [x < \Omega \wedge \Phi(x) > q]$$

The right-hand side of the equivalence is a $\Sigma^0_1$ predicate, hence the left cut of $\Omega^*$ is c.e., from which it follows that $\Omega^*$ is left-c.e.

Let us make a few remarks. First if a non-computable c.e. set is Martin-Löf random with respect to a computable probability measure, then it must be Turing complete. Indeed, by Demuth's Theorem, such a real must be Turing equivalent to a real that is Martin-Löf random for Lebesgue measure and Kučera [Kuč85] proved that a c.e. real that can compute a Martin-Löf random real must necessarily be Turing complete.

The family of c.e. sets that are random for some computable probability measure is therefore not downwards closed in the Turing degrees. However, this family is closed downwards in the $tt$-degrees by Demuth's Theorem: given a $tt$-functional $\Phi$ and a c.e. set that is random with respect to a computable measure $\mu$, if $\Phi(c)$ is c.e. then it is either computable or Turing complete, and in both cases, it will be random with respect to the measure induced by $(\mu, \Phi)$. It is thus natural to consider whether the family of c.e. random sets forms a $tt$-ideal. As we now show, they do not.

**Proposition 6.9** *The c.e. random reals do not form a tt-ideal.*

**PROOF.** Let $a$ be a left-c.e., Turing incomplete, real and $x_1$ a left-c.e. random real. Set $x_2 = x_1 + a$ and notice that $x_2$ is left-c.e. and random as the sum of a random left-c.e. real and a left-c.e. real [DH10, Chapter 8]. Now convert $x_1$ and $x_2$ into c.e. reals via $T$: $y_1 = T(x_1)$ and $y_2 = T(x_2)$. Then both $y_1$ and $y_2$ are c.e. and random with respect to the measure $\mu$ induced by $T$.

Since $T$ is a total computable map, its range is a $\Pi^0_1$ class, call it $\mathcal{C}$. Since $T$ is one-to-one, the function $T^{-1}$ is Turing-computable on its domain $\mathcal{C}$ (indeed for all $z \in \mathcal{C}$, the set $\{x : T(x) = z\}$ is a $\Pi^0_1(z)$ class containing only one element, hence that element can be computably found when $z$ is given). It is well-known that a partial functional defined on a $\Pi^0_1$ class can be extended to a total functional. Then let $S$ be a *tt*-functional which is an extension of $T^{-1}$ to the entire space $2^\omega$.

Now, suppose that the join $y_1 \oplus y_2$ is random with respect to some computable measure $\nu$. Consider the functional $\Psi$ defined by $\Psi(z_1 \oplus z_2) = |S(z_1) - S(z_2)|$. This is a *tt*-functional, and $\Psi(y_1 \oplus y_2) = a$. By the Conservation of Martin-Löf randomness, this means that $a \in \mathsf{MLR}_{\nu_\Psi}$. This is a contradiction since by the discussion above, an incomplete (left-)c.e. real cannot be Martin-Löf random w.r.t. any computable measure.

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## References

[BM09] Laurent Bienvenu and Wolfgang Merkle. Constructive equivalence relations on computable probability measures. *Ann. Pure Appl. Logic*, 160(3):238–254, 2009.

[Dem88] Osvald Demuth. Remarks on the structure of tt-degrees based on constructive measure theory. *Comment. Math. Univ. Carolin.*, 29(2):233–247, 1988.

[DH10] Rodney Downey and Denis Hirschfeldt. *Algorithmic randomness and complexity.* Theory and Applications of Computability. Springer, 2010.

[Fra08] Johanna Franklin. Hyperimmune-free degrees and Schnorr triviality. *Journal of Symbolic Logic*, 73:999–1008, 2008.

[Kau91] Steven M. Kautz. Degrees of random sets. 1991.

[KHMS11] Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. Kolmogorov complexity and the recursion theorem. *Transactions of the American Mathematical Society*, 363(10):5465–5480, 2011.

[KL10] Bart Kastermans and Steffen Lempp. Comparing notions of randomness. *Theoretical Computer Science*, 411(3):602–616, 2010.

[Kuč85] Antonin Kučera. Measure, $\Pi^0_1$ classes, and complete extensions of PA. *Lecture Notes in Mathematics*, 1141:245–259, 1985.

[Kur81] Stuart Kurtz. *Randomness and genericity in the degrees of unsolvability.* PhD dissertation, University of Illinois at Urbana, 1981.

[LZ70] L. A. Levin and A. K. Zvonkin. The complexity of finite objects and the basing of the concepts of information and randomness on the theory of algorithms. *Uspehi Mat. Nauk*, 25(6(156)):85–127, 1970.

[MMN+06] Wolfgang Merkle, Joseph S. Miller, André Nies, Jan Reimann, and Frank Stephan. Kolmogorov-Loveland randomness and stochasticity. *Annals of Pure and Applied Logic*, 138(1-3):183–210, 2006.

[MSU98] Andrei A. Muchnik, Alexei Semenov, and Vladimir Uspensky. Mathematical metaphysics of randomness. *Theoretical Computer Science*, 207(2):263–317, 1998.

[Nie09] André Nies. *Computability and randomness*. Oxford Logic Guides. Oxford University Press, 2009.

[NST05] André Nies, Frank Stephan, and Sebastiaan Terwijn. Randomness, relativization and Turing degrees. *Journal of Symbolic Logic*, 70:515–535, 2005.

[RS07] Jan Reimann and Theodore A. Slaman. Probability measures and effective randomness. *Preprint*, 2007.

[RS08] Jan Reimann and Theodore A. Slaman. Measures and their random reals. *Preprint*, 2008.

[SF77] C.-P. Schnorr and P. Fuchs. General random sequences and learnable sequences. *J. Symbolic Logic*, 42(3):329–340, 1977.

[Soa87] Robert I. Soare. *Recursively enumerable sets and degrees*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. A study of computable functions and computably generated sets.

## A Appendix

**PROOF of Lemma 2.6.** We proceed inductively as follows: First,

$$\mu_\Phi(\emptyset) = \mu(\Phi^{-1}(\llbracket\emptyset\rrbracket)) = \mu(\Phi^{-1}(\mathrm{dom}(\Phi))) = 1,$$

since $\Phi$ is almost total. Now suppose that $\mu_\Phi(\sigma)$ is computable. Then $\mu_\Phi(\sigma 0)$ and $\mu_\Phi(\sigma 1)$ are both approximable from below, and since $\mu_\Phi(\sigma) = \mu_\Phi(\sigma 0) + \mu_\Phi(\sigma 1)$, it follows that both $\mu_\Phi(\sigma 0)$ and $\mu_\Phi(\sigma 1)$ are approximable from above. Thus, both are computable.

For the second part, let $\varphi$ be a computable function that bounds the use of $\Phi$, i.e. if $\Phi(x) = y$, then for every $n \in \omega$, $\Phi^{x\upharpoonright\varphi(n)} \succeq y\upharpoonright n$. Without loss of generality, we can assume that if $|\sigma| = n$ and $|\tau| < \varphi(n)$, then $\Phi^\tau \not\succeq \sigma$. If we define

$$\mathsf{Pre}_\Phi(\sigma) := \{\tau \in 2^{<\omega} : \Phi^\tau \succeq \sigma \ \wedge \ (\forall \tau' \preceq \tau)\Phi^{\tau'} \not\succeq \sigma\},$$

(so that $\llbracket\mathsf{Pre}_\Phi(\sigma)\rrbracket = \Phi^{-1}(\llbracket\sigma\rrbracket)$), it follows that

$$\mathsf{Pre}_\Phi(\sigma) = \{\tau \in 2^{\varphi(|\sigma|)} : \Phi^\tau \succeq \sigma\}$$

and thus

$$\mu_\Phi(\sigma) = \mu(\Phi^{-1}(\llbracket\sigma\rrbracket)) = \mu\Big(\bigcup_{\tau \in \mathsf{Pre}_\Phi(\sigma)} \llbracket\tau\rrbracket\Big) = \sum_{\tau \in \mathsf{Pre}_\Phi(\sigma)} \mu(\tau),$$

which is $\mathbb{Q}_2$-valued because $\mu$ is $\mathbb{Q}_2$-valued and $\mathsf{Pre}_\Phi(\sigma)$ is finite. Moreover, since we can find, effectively in $\sigma$, the index for $\mathsf{Pre}_\Phi(\sigma)$ as a finite set, it follows that $\mu_\Phi$ is a computable function from $2^{<\omega}$ to $\mathbb{Q}_2$, and thus is exactly computable.

**Remark A.1** *In the proof of Theorem 3.2 below, we will have to be careful with the enumeration of our Martin-Löf tests, and so we will ensure that these tests have nice presentations. Recall that a set $S \subseteq 2^{<\omega}$ is prefix-free if for every $\sigma, \tau \in S$, if $\sigma \preceq \tau$, then $\sigma = \tau$. Then given a Martin-Löf test $\{\mathcal{U}_i\}_{i\in\omega}$, we will say that a uniformly computable sequence $\{S_i\}_{i\in\omega}$ of subsets of $2^{<\omega}$ is a prefix-free presentation of $\{\mathcal{U}_i\}_{i\in\omega}$ if we have $\mathcal{U}_i = [S_i]$ for every $i \in \omega$.*

**PROOF of Theorem 3.2.** Suppose that $\Phi(x) \notin \mathsf{MLR}_{\mu_\Phi}$; we will show that $x \notin \mathsf{MLR}_\mu$. Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a $\mu_\Phi$-Martin-Löf test such that $\Phi(x) \in \bigcap_{i\in\omega} \mathcal{U}_i$. We define a $\mu$-Martin-Löf test $\{\mathcal{V}_i\}_{i\in\omega}$ containing $x$ as follows. First, let $\{S_i\}_{i\in\omega}$ be a prefix-free presentation of $\{\mathcal{U}_i\}_{i\in\omega}$. Then we define, for each $i \in \omega$,

$$P_i = \bigcup_{\sigma \in S_i} \mathsf{Pre}_\Phi(\sigma).$$

Note that since $S_i$ is prefix-free, for distinct $\sigma_1, \sigma_2 \in S_i$, $\mathsf{Pre}_\Phi(\sigma_1) \cap \mathsf{Pre}_\Phi(\sigma_2) = \emptyset$, and so $\bigcup_{\sigma\in S_i} \mathsf{Pre}_\Phi(\sigma)$ is a disjoint union. Hence

$$\mu(\llbracket P_i\rrbracket) = \mu\Big(\bigcup_{\sigma\in S_i} \llbracket\mathsf{Pre}_\Phi(\sigma)\rrbracket\Big) = \sum_{\sigma\in S_i} \mu(\llbracket\mathsf{Pre}_\Phi(\sigma)\rrbracket) = \sum_{\sigma\in S_i} \mu(\Phi^{-1}(\llbracket\sigma\rrbracket)) = \mu_\Phi(\mathcal{U}_i).$$

Now if we set $\mathcal{V}_i := \llbracket P_i\rrbracket$ for each $i$, we have $\mu(\mathcal{V}_i) = \mu_\Phi(\mathcal{U}_i)$ for each $i$. In addition, since the collection $\{\mathcal{V}\}_{i\in\omega}$ is definable uniformly from $\{\mathcal{U}\}_{i\in\omega}$, it follows that $\{\mathcal{V}\}_{i\in\omega}$ is a $\mu$-Martin-Löf test. Lastly, we must verify that $x \in \bigcap_{i\in\omega} \mathcal{V}_i$. For each $i$, since $\Phi(x) \in \mathcal{U}_i$, there is some $\sigma \in S_i$ and some least $n \in \omega$ such that $\Phi^{x\restriction n} \succeq \sigma$. Thus $x\restriction n \in \mathrm{Pre}_\Phi(\sigma)$, and so it follows that $x\restriction n \in P_i$ and $x \in \mathcal{V}_i$.

**PROOF of Theorem 3.4.** The key observation in Kautz's proof is that for a given computable measure $\mu$, almost every $x \in [0,1]$ has a binary representation given in terms of $\mu$, which we will refer to as its $\mu$-representation, denoted by Kautz as $\mathrm{seq}_\mu(x)$. Using this $\mu$-representation, we will define $\Phi$ so that $\Phi$ maps $x$ (considered as an infinite binary sequence) to $\mathrm{seq}_\mu(x)$ (also considered as an infinite binary sequence). We should emphasize that even though the correspondence between [0,1] and $2^\omega$ is not one-to-one (since rationals can be represented as sequences with cofinitely many 0s or cofinitely many 1s), in order to prove the theorem, we only need to consider the value that $\Phi$ takes on non-computable, and hence non-rational, reals.

To compute the $\mu$-representation of $x \in [0,1]$, we make use of what we call a $\mu$-partition of [0,1]. A *$\mu$-partition of* $[0,1]$ *at level* $n$ is a collection of $k = 2^n$ closed intervals $I_{\sigma_0}, I_{\sigma_1}, \dots I_{\sigma_{k-1}}$ such that

(1) $\sigma_0, \sigma_1, \dots, \sigma_{k-1}$ is a listing of all strings of length $n$ in lexicographical ordering,
(2) $\bigcup_{i=0}^{k-1} I_{\sigma_i} = [0,1]$,
(3) $\sup I_{\sigma_i} = \inf I_{\sigma_{i+1}}$ for $0 \leq i \leq k-2$, and
(4) $\mu(\sigma_i) = \lambda(I_{\sigma_i})$ for $0 \leq i \leq k-1$.

We further require that the $\mu$-partition of level $n$ is compatible with the $\mu$-partition of level $n+1$ for every $n$, so that given a string $\sigma$ of length $n$, we have

$$I_\sigma = I_{\sigma 0} \cup I_{\sigma 1}.$$

Now, given a real $x \in [0,1]$ we can compute its $\mu$-representation $\mathrm{seq}_\mu(x)$ as follows. To determine the first bit of $\mathrm{seq}_\mu(x)$, we consider the $\mu$-partition of [0,1] at level 1, $I_0 \cup I_1$. Given that $\mu$ is computable but not necessarily exactly computable, we may have to approximate $I_0$ and $I_1$ until we see that $x \in I_0$ or $x \in I_1$, which will occur as long as $x$ is not the right endpoint of $I_0$ (we omit the details). If $x \in I_0$, the first bit of $\mathrm{seq}_\mu(x)$ is a 0, and if $x \in I_1$, the first bit of $\mathrm{seq}_\mu(x)$ is a 1. Having determined the first $n$ bits of $\mathrm{seq}_\mu(x)$ by finding $\sigma$ such that $|\sigma| = n$ and $x \in I_\sigma$, we determine whether $x \in I_{\sigma 0}$ or $x \in I_{\sigma 1}$ (where $I_{\sigma 0}$ and $I_{\sigma 1}$ are given by the $\mu$-partition of [0,1] at level $n+1$), and output a 0 or 1 accordingly, as in base case described above.

Thus, if $x$ is not an endpoint of $I_\sigma$ for any $\sigma \in 2^{<\omega}$, then $\mathrm{seq}_\mu(x)$ is the unique real $y \in 2^\omega$ such that $x \in I_{y \upharpoonright n}$ for every $n$. Clearly, then $\Phi$ is almost total, and we have that

$$\Phi^{-1}([\![\sigma]\!]) = \{x : \Phi(x) \succ \sigma\} = I_\sigma,$$

so that

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}([\![\sigma]\!])) = \lambda(I_\sigma) = \mu(\sigma).$$

In the case that $\mu$ is atomless, we have moreover that for every $y \in 2^\omega$, $\lim_{n\to\infty} \lambda(I_{y\upharpoonright n}) = 0$, which implies that there is a unique $x$ such that $\bigcap_{n\in\omega} I_{y\upharpoonright n} = \{x\}$. Thus, if $\Phi(x_1) = \Phi(x_2)$, we must have $x_1 = x_2$. Next, if $\mu$ is positive, then for every $\sigma \in 2^{<\omega}$, $\lambda(I_\sigma) > 0$, which means that $\Phi^{-1}(\sigma)$ is non-empty for every $\sigma \in 2^{<\omega}$. Thus, given $y \in 2^\omega$, since

$$\Phi^{-1}([\![y\upharpoonright n]\!]) \supseteq \Phi^{-1}([\![y\upharpoonright(n+1)]\!])$$

for every $n$ and each is non-empty, there is some $x$ such that

$$x \in \bigcap_{n \in \omega} \Phi^{-1}(\llbracket y \restriction n \rrbracket).$$

Lastly, in the case that $\mu$ is both atomless and positive, then since $\Phi$ is one-to-one, it has an inverse $\Phi^{-1}$. Since $\Phi$ is onto up to a set of measure zero, it follows that $\Phi^{-1}$ is almost total. Given $y \in 2^\omega$ in the range of $\Phi$, i.e. $\Phi(x) = y$ for some $x \in 2^\omega$, then $\Phi^{-1}(y)$ can be computed by successively computing $\Phi^{-1}(\llbracket y \restriction n \rrbracket)$ for each $n$ and then intersecting these sets. More specifically, since

$$\bigcap_{n \in \omega} \Phi^{-1}(\llbracket y \restriction n \rrbracket) = \{x\},$$

for each $i$, we will eventually find some $n_i$ such that

$$z \in \bigcap_{n \leq n_i} \Phi^{-1}(\llbracket y \restriction n \rrbracket) \Rightarrow z \restriction i = x \restriction i.$$

Thus we will have $(\Phi^{-1})^{y \restriction n_i} \succeq x \restriction i$ for every $i$.

**Proposition A.1** *The following are equivalent.*

*(i) $x$ is complex in the sense of Definition 5.2.*
*(ii) There exists a computable order function $h$ such that $C(x \restriction n) \geq h(n)$ for all $n$, $C$ denoting plain Kolmogorov complexity.*

Item (ii) corresponds to the original definition of complex reals by Kjos-Hanssen et al.

**PROOF.** $(i) \to (ii)$ is trivial as $Km \leq 2C$. For the reverse direction, we use the following result of Kjos-Hanssen et al: a real satisfies (ii) if and only if it *wtt*-computes a sequence of strings $(\sigma_n)$ such that $C(\sigma_n) \geq n$. Now suppose that $x$ is complex, and therefore *wtt*-computes such a sequence $\sigma_n$. Let $\varphi$ be a computable bound on the use of this reduction. We have the following inequalities:

$$Km(x \restriction \varphi(n)) \geq^+ C(\sigma_n) - 2 \log n \geq n - 2 \log n \geq^+ n/2$$

The second inequality is true by definition of $\sigma_n$. To see that the first one holds, let $p$ be the shortest $Km$-description of $x \restriction \varphi(n)$. Using $p$, one can compute an extension $\tau$ of $x \restriction \varphi(n)$. Then, specifying $n$ (for a cost of $2 \log n + O(1)$ bits), one can retrieve $x \restriction \varphi(n)$ and therefore compute $\sigma_n$. Since $Km$ is monotonic, it follows that

$$Km(x \restriction n) \geq^+ \varphi^{-1}(n)/2$$

and the right-hand side is a computable order function.

**PROOF of Proposition 6.4** The idea behind the proof is to define a non-decreasing $tt$-functional $\Theta$ such that $\mu_\Theta$ is a generalized Bernoulli measure, i.e. such that for every $n$, there is some $p_n \in [0,1]$ such that

$$p_n = \frac{\mu_\Theta(\sigma 0)}{\mu_\Theta(\sigma)}$$

for every $\sigma \in 2^{<\omega}$ of length $n$. Moreover, we will define $\Theta$ so that

$$|p_n - 1/2|^2 \leq 2^{-|\sigma|}$$

for every $n \in \omega$. Lastly, we would like to define $\Theta$ in such a way that the resulting values $p_n$ will always be contained in some fixed interval $[\epsilon, 1-\epsilon]$ for $\epsilon \in (0, \frac{1}{2})$; such measures are called *strongly positive*. Now by the effective version of Kakutani's Theorem (see, for instance, [BM09]), given two computable, strongly positive, generalized Bernoulli measures $\mu_1$ (with associated values $p_1, p_2, \dots$) and $\mu_2$ (with associated values $q_1, q_2, \dots$) such that

$$\sum_{i=1}^{\infty} |p_i - q_i|^2 < \infty,$$

it follows that $\mathsf{MLR}_{\mu_1} = \mathsf{MLR}_{\mu_2}$. Thus, if we can define such $\Theta$ satisfying the given conditions, then we will have

$$\sum_{i=1}^{\infty} |p_i - 1/2|^2 < \infty,$$

and hence $\mathsf{MLR}_{\mu_\Theta} = \mathsf{MLR}$.

To define $\Theta$, we sketch the main idea and leave the details to the reader. To define $p_1$, we look for a finite, prefix-free collection of strings $\{\sigma_1, \dots, \sigma_k\}$ such that

$$I_{\sigma_1} \cup \dots \cup I_{\sigma_k} = [0, \frac{1}{2} - \epsilon_1],$$

for some $\epsilon_1 < \frac{1}{2}$, where $I_\sigma$ is as defined in the proof of Theorem 3.4 (we can find such a collection effectively because $\mu$ is atomless). Then we define $\Theta$ so that extensions of each $\sigma_i$ is mapped to extensions of 0 (and reals that extend none of the $\sigma_i$'s are mapped to extensions of 1). Thus $p_1 = \sum_{i \leq k} \mu(\sigma_i)$.

Now we repeat this procedure, partitioning the intervals $[0, \frac{1}{2} - \epsilon_1]$ and $[\frac{1}{2} - \epsilon_1, 1]$ each into two intervals, each of which is determined by a finite, prefix-free collection of strings, just as we partitioned the interval [0,1] above, but we must make sure that ratios of the sizes of the components of each partition is the same, i.e. the left component of each is $p_2$ times the length of the given interval, where $p_2$ is within $\frac{1}{4}$ of $\frac{1}{2}$. In so doing, we will get four collections of strings, extensions of which will be mapped to extensions of 00,01,10,11 (depending on which of the four partitions the sequences belong to). Continuing this procedure, we will eventually define $\Theta$ with the desired properties.