Demuth's Work on Randomness and Analysis

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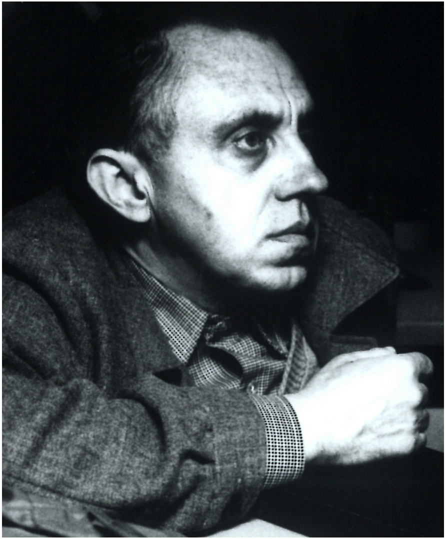

Osvald Demuth

Introducing Demuth

- Born in Prague in 1936.
- Graduated in 1959 from Charles University in Prague with the equivalent of a master's degree.
- Studied constructive mathematics with Markov from 1959 until 1964, when he defended his doctoral thesis.
- Returned to Charles University, working mostly in isolation until his death in 1988.

Outline of the Talk

1. The Markov school of constructive mathematics
2. Demuth randomness
3. Randomness and differentiability
4. Reducibilities from constructive analysis
5. *tt*-reducibility, randomness, and semigenericity

1. The Markov school of constructive mathematics

The aim of constructive mathematics

> *The aim was to find [a] foundation for mathematics that would be as simple and secure as possible and would be free from too far reaching idealisations, foundations in which the concept of effectivity would be the principal.*

"Remarks on Constructive Mathematical Analysis"
Demuth, Kučera 1979

Historical motivation for constructive mathematics

[F]rom the historical point of view, the development of mathematics was substantially influenced by applications of mathematics where solutions of problems consisted, de facto, in transformation of particular information coded by words.

To put it shortly, the main subject of CM is the study of possibilities of algorithmical transformation of coded information about mathematical objects.

"Remarks on Constructive Mathematical Analysis"
Demuth, Kučera 1979

An unorthodox constructivism? I

It should be noted that we are interested, owing to the natural connection between concepts of constructive mathematical analysis and arithmetical predicates, only in computability relative to jumps of the empty set.

It is known from the results of E.M. Gold and [H.] Putnam that the $\emptyset^{(n)}$-PRFs ($1 \leq n$) can be represented on the basis of recursive functions by means of non-effective limits...

An unorthodox constructivism? II

Without leaving [the] constructive program concerning effective processes we improve, by the use of relative computability, our ability to handle effective procedures. The advantage of the improvement consists in both substantial simplification and clearness of formulations.

"Remarks on Constructive Mathematical Analysis"
Demuth, Kučera 1979

2. Demuth randomness

The definition of Demuth randomness

Definition

- A Demuth test is a sequence of effectively open classes $(\mathcal{U}_i)_{i \in \mathbb{N}}$ such that
  (i) $\lambda(\mathcal{U}_n) \leq 2^{-n}$ for every n, and
  (ii) there is a function $f : \mathbb{N} \to \mathbb{N}$ with $f \leq_{\mathrm{wtt}} \emptyset'$ such that $\mathcal{U}_n = [\![W_{f(n)}]\!]$ for every n.
- $Z \in 2^\omega$ passes the Demuth test $(\mathcal{U}_i)_{i \in \mathbb{N}}$ if $Z \notin \mathcal{U}_n$ for almost every n.
- $Z \in 2^\omega$ is Demuth random if Z passes every Demuth test.

Properties of Demuth randomness, I

Demuth proved the following facts about Demuth random (DemR) sequences:

- Every $Z \in \mathsf{DemR}$ is generalized low (in fact, he proved $Z' \equiv_{\emptyset'\text{-tt}} Z$).
- Every $Z \in \mathsf{DemR}$ has hyperimmune degree.

Properties of Demuth randomness, II

- There is a $\emptyset'$-computable function g such that for every $Z \in \mathsf{DemR}$ and for every partial $f \leq_T Z$,

$$f(n) \leq g(n)$$

for almost every n (and thus $\emptyset'$ is uniformly almost everywhere dominating).

- For every $Z \in \mathsf{DemR}$ and $A \in \mathsf{MLR}$, $A \leq_T Z$ implies $A \in \mathsf{DemR}$.

Demuth jump inversion

Theorem (Demuth 1988)

Let $B, C \in 2^\omega$.

For any $\mathcal{E} \subseteq 2^\omega$ of B-measure zero, there is $A \notin \mathcal{E}$ such that

- $A \leq_{T(B)} C$ *and*
- $C \leq_{T(B)} A$.

Corollary

For any $C \geq_T \emptyset'$ there is some $A \in \mathsf{DemR}$ such that $A' \equiv_T C$.

Demuth and density

Theorem (Demuth 1988)

For every $Z \in \mathsf{DemR}$ and every Π^0_1 $\mathcal{P}$ class containing Z, for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that for all $y \geq k$,

$$\lambda(\mathcal{P} \cap [\![Z \upharpoonright y]\!]) \geq (1 - 2^{-n})\lambda([\![Z \upharpoonright y]\!]).$$

Corollary

Every Demuth random is a density-one point.

Theorem (Bienvenu, Greenberg, Kučera, Nies, Turetsky)

Every Oberwolfach random is a density-one point.

2. Randomness and differentiability

“Constructive” real numbers

A real $r \in [0,1]$ is constructive if

- there is a computable sequence of rationals $(q_n)_{n\in\mathbb{N}}$ converging to r, and
- a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every n and every $k \geq f(n)$,
$$|q_{f(n)} - q_k| \leq 2^{-n}.$$

Nowadays, we refer to such real numbers as computable.

Genuinely constructive real numbers

Actually, this isn't the definition Demuth used.

For him, a constructive real number r is given by a pair of natural numbers (e_1, e_2), where

- e_1 is the index for a computable sequence of rationals $(q_n)_{n\in\mathbb{N}}$ converging to r, and
- e_2 is the index of the modulus function f (which Demuth calls the regulator of fundamentality of the sequence $(q_n)_{n\in\mathbb{N}}$).

Markov computable functions

A constructive function $f : \mathbb{R}_c \to \mathbb{R}_c$ is an effective procedure such that

- for any constructive real number x, if $f(x)\downarrow$, then $f(x)$ is constructive, and
- for any constructive x, y, if $f(x)\downarrow$ and $x = y$, then $f(y)\downarrow$ and $f(x) = f(y)$.

“Standard” computable functions

The standard definition of a computable real-valued function:

- $f : \mathbb{R} \to \mathbb{R}$ is computable if there is some Turing functional that maps every Cauchy name for $x \in \mathrm{dom}(f)$ to a Cauchy name of $f(x)$.

How does this relate to Markov computable functions?

“Standard” computable functions vs. Markov computable functions

standard computable functions $\Rightarrow$ uniformly continuous

Markov computable functions $\not\Rightarrow$ uniformly continuous

However, every uniformly continuous Markov computable function with a computable modulus of uniform continuity can be obtained as the restriction of a “standard” computable real-valued function.

Demuth's contribution

Demuth was the first to recognize the deep connection between randomness and differentiability.

Theorem (Demuth 1975)

Every Markov computable function of bounded variation is "differentiable" at each Martin-Löf random real.

Upper and lower derivatives

For $f : \mathbb{R} \to \mathbb{R}$, the slope at a pair $a, b \in \mathrm{dom}(f)$ is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}.$$

If $z \in \mathrm{dom}(f)$ then

$$\overline{D}f(z) = \limsup_{h \to 0} S_f(z, z+h)$$

and

$$\underline{D}f(z) = \liminf_{h \to 0} S_f(z, z+h).$$

If $\overline{D}f(z) = \underline{D}f(z) < \infty$, then $f'(z)$ exists.

Pseudo-differentiability

How does one take the derivative of a real at a Markov computable function?

Let $I_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$.

For a Markov computable f and $z \in [0,1]$ define

$$\widetilde{D}f(z) = \limsup_{h \to 0^+} \{S_f(a,b) : a, b \in I_{\mathbb{Q}} \ \wedge \ a \leq x \leq b \ \wedge \ 0 < b - a \leq h\}$$

and

$$\underset{\sim}{D}f(z) = \liminf_{h \to 0} \{S_f(a,b) : a, b \in I_{\mathbb{Q}} \ \wedge \ a \leq x \leq b \ \wedge \ 0 < b - a \leq h\}.$$

Recent results

Many more results of this flavor have been established recently. For example:

Theorem (Brattka, Miller, Nies)

1. $z \in [0, 1]$ *is Martin-Löf random if and only if every Markov computable function of bounded variation is differentiable at* z*.*
2. $z \in [0, 1]$ *is Martin-Löf random if and only if every computable function of bounded variation is differentiable at* z*.*
3. $z \in [0, 1]$ *is computably random if and only if every nondecreasing computable function is differentiable at* z*.*

Denjoy random reals and the Denjoy alternative

Definition

$z \in [0,1]$ is Denjoy random if for no Markov computable function f do we have $\underset{\sim}{D}f(z) = \infty$.

Definition

A partial function $f :\subseteq [0,1] \to \mathbb{R}$ with dense domain satisfies the Denjoy alternative at $z \in [0,1]$ if

$$\text{either } f'(z) \text{ exists, or } \widetilde{D}f(z) = \infty \text{ and } \underset{\sim}{D}f(z) = -\infty.$$

Demuth and Denjoy

Theorem (Demuth 1988)

If $z \in [0,1]$ is Denjoy random, then for every computable $f : [0,1] \to \mathbb{R}$, the Denjoy alternative holds at f.

Theorem (Demuth 1983)

Let $z \in [0,1]$ be Demuth random. Then the Denjoy alternative holds at z for every Markov computable function.

Theorem (Demuth 1976)

There is a Markov computable function f such that the Denjoy alternative fails at some $z \in \mathsf{MLR}$.

What we currently know about Denjoy randoms

Theorem (Bienvenu, Hölzl, Miller, Nies)

The following are equivalent for $z \in [0, 1]$.

(i) *z is Denjoy random.*

(ii) *z is computably random.*

(iii) *For every computable $f : [0, 1] \to \mathbb{R}$, the Denjoy alternative holds at z.*

What we currently know about the Denjoy alternative

Theorem (Brattka, Miller, Nies)

Let $z \in [0,1]$ be such that the Denjoy alternative holds at z for every Markov computable function. Then z is computably random.

Theorem (Bienvenu, Hölzl, Miller, Nies)

Let $z \in [0,1]$ be difference random. Then the Denjoy alternative holds at z for every Markov computable function.

3. Reducibilities from constructive analysis

Bridging computability theory and constructive mathematics

Demuth proved a number of results connecting truth-table reducibility and various reducibilities from constructive analysis.

To establish these bridge results, Demuth restricted the class of constructive functions f to those that are constant on $(-\infty, 0]$ and $[1, \infty)$.

Demuth called these functions *c*-functions.

Extending c-functions to classical functions

Let $g : \mathbb{R}_c \to \mathbb{R}_c$ be a c-function.

Then $R[g]$ is the classical function that is the maximal (i.e., has the largest domain) continuous (on its domain) extension of g.

That is, for each non-computable $r \in [0, 1]$, if there is some $a \in \mathbb{R}$ such that

$$\lim_{x \to r^-} g(x) = \lim_{x \to r^+} g(x) = a,$$

then we set $R[g](r) = a$. Otherwise, $R[g](r)$ is undefined.

f-reducibility

By passing from g to $R[g]$, we can define a reducibility on pairs of reals:

Let g be a c-function. Given $\alpha, \beta \in [0,1]$, α is f-reducible to β via g, denoted $\alpha \leq_f \beta$, if

$$R[g](\beta) = \alpha.$$

A few more reducibilities

Let $\alpha, \beta \in [0, 1]$.

1. α is $\emptyset$-ucf reducible to β, denoted $\alpha \leq_{\emptyset\text{-ucf}} \beta$, if α is f-reducible to β via a c-function g that is $\emptyset$-uniformly continuous.
2. α is $\emptyset'$-ucf reducible to β, denoted $\alpha \leq_{\emptyset'\text{-ucf}} \beta$ if α is f-reducible to β via a c-function g that is $\emptyset'$-uniformly continuous.
3. α is mf reducible to β, denoted $\alpha \leq_{\mathrm{mf}} \beta$ if α is f-reducible to β via a c-function g that is monotonically increasing.

Bi-infinite sets

For Demuth's purposes, reals that have two different binary representations are problematic.

To rule these out, he introduces the following definitions.

Definition

1. A c.e. set $S \subseteq 2^{<\omega}$ is a finite set cover if for every finite set Z, one of its two binary representations is covered by $[\![S]\!]$.
2. $z \in [0, 1]$ is strongly bi-infinite if z is bi-infinite and there is some finite set cover S such that $z \notin [\![S]\!]$.
3. $z \in [0, 1]$ is weakly bi-infinite if z is bi-infinite but not strongly bi-infinite.

tt-reducibility and $\emptyset$-ucf-reducibility

Theorem (Demuth 1988)

1. *For any $\emptyset$-uniformly continuous c-function $f : [0,1] \to [0,1]$, for any $A, B \in 2^\omega$ such that B is strongly bi-infinite,*

$$B \leq_{\emptyset\text{-}ucf} A \text{ via } f \text{ if and only if } B \leq_{tt} A.$$

2. *For any tt-functional Φ, there is a $\emptyset$-uniformly continuous c-function $f : [0,1] \to [0,1]$ such that for any $A, B \in 2^\omega$ such that A is strongly bi-infinite and B is bi-infinite,*

$$B \leq_{\emptyset\text{-}ucf} A \text{ via } f \text{ if and only if } B \leq_{tt} A.$$

$\emptyset'$-*tt*-reducibility and $\emptyset'$-*ucf*-reducibility

Theorem (Demuth 1988)

1. *For any $\emptyset'$-uniformly continuous c-function $f : [0,1] \to [0,1]$, for any $A, B \in 2^\omega$ such that B is strongly bi-infinite relative to $\emptyset'$,*

$$B \leq_{\emptyset'\text{-}ucf} A \text{ via } f \text{ if and only if } B \leq_{\emptyset'\text{-}tt} A.$$

2. *For any tt-functional Φ, there is a $\emptyset$-uniformly continuous c-function $f : [0,1] \to [0,1]$ such that for any $A, B \in 2^\omega$ such that A is strongly bi-infinite relative to $\emptyset'$ and B is bi-infinite,*

$$B \leq_{\emptyset'\text{-}ucf} A \text{ via } f \text{ if and only if } B \leq_{\emptyset'\text{-}tt} A.$$

Why care about $\leq_{\emptyset'\text{-ucf}}$?

Theorem (Demuth)

Let f be a c-function. Then the following are equivalent.

1. *f is classically uniformly continuous.*
2. *f is $\emptyset'$-uniformly continuous.*
3. *$R[f]$ is defined at all Δ^0_3 reals.*
4. *$R[f]$ is defined on all reals.*

tt-reducibility and mf-reducibility

Theorem (Demuth 1988)

Let f be a non-decreasing c-function such that $f(0) = 0$ and $f(1) = 1$. Then the following are equivalent.

1. *For any $A, B \in 2^\omega$,*
$$B \leq_{tt} A \text{ implies } B \leq_{mf} A \text{ via } f.$$
2. *There is a computable $g : \mathbb{Q}_2 \to \mathbb{Q}_2$ such that for every $a \in \mathbb{Q}_2 \cap [0, 1]$,*
$$f(g(a)) = a.$$

4. *tt*-reducibility, randomness, and semigenericity

tt-reductions and randomness

Theorem (Demuth 1988)

If B is non-computable and tt-reducible to $A \in \mathsf{MLR}$, then there is some $C \in \mathsf{MLR}$ such that

$$B \leq_{tt} C \leq_{T} B.$$

Recent proofs of this result are given in terms of computable measures.

Demuth's proof makes use of the distribution function of the measure induced by the initial *tt*-functional, which he considers in terms of *mf*-reducibility.

C.e. random sets

Theorem (Demuth 1987)

There is a tt-degree containing both a c.e. set and a Martin-Löf random set.

Corollary

There is some c.e. set $S \in 2^\omega$ that is Martin-Löf random with respect to some computable measure.

Semigenericity

Definition

$Z \in 2^\omega$ is semigeneric if for every Π^0_1 class $\mathcal{P}$ such that $Z \in \mathcal{P}$, there is some computable $C \in \mathcal{P}$.

Some facts about semigenericity:

- Semigenerics are closed downwards under $\leq_{tt}$.
- Every weakly 1-generic is semigeneric.
- All hyperimmune and all co-hyperimmune sets are semigeneric.

Semigenericity and randomness

In general, the following does not hold:

If B is non-computable and *tt*-reducible to $A \in \mathsf{MLR}$, then there is some $C \in \mathsf{MLR}$ such that

$$B \leq_{tt} C \leq_{wtt} B.$$

Theorem (Bienvenu, Porter)

There is some $A \in \mathsf{MLR}$ and a tt-functional Φ such that $\Phi(A)$ is non-computable and cannot wtt-compute any Martin-Löf random.

Remarkably, $\Phi(A)$ from this theorem is semigeneric.

Random and semigeneric hyperimmune-free degrees

No semigeneric can *tt*-compute a Martin-Löf random.

Thus, a semigeneric hyperimmune-free degree contains no Martin-Löf randoms, and a Martin-Löf random hyperimmune-free degree contains no semigenerics.

Demuth also showed there are hyperimmune-free degrees that are neither Martin-Löf random nor semigeneric (consider the degree of $A \oplus B$ for $A \in \mathsf{MLR}$ and HIF and B semigeneric and HIF).

Randomness, semigenericity, and Denjoy randoms

Theorem (Demuth 1990)

1. *Every non-*MLR *Denjoy random real is high.*
2. *Every high degree computes a semigeneric Denjoy random real.*
3. *There is a minimal Turing degree containing a semigeneric Denjoy random real.*
4. *No non-*MLR X *tt-below some* $Y \in$ MLR *can be a Denjoy random real.*
5. *Every semigeneric Denjoy random real is tt-reducible to a Denjoy set that is neither semigeneric nor Martin-Löf random.*

For more details...

"Demuth's Path to Algorithmic Randomness" by André Nies and Antonín Kučera is a fairly thorough discussion of Demuth's work relating randomness and analysis.

An extended version of "Demuth's Path", including

- some additional material on Demuth randomness discussed in this talk,
- the reducibilities from constructive analysis,
- interactions between *tt*-reducibility, randomness, and semigenericity,
- and a bit of Demuth's views on constructive mathematics,

will be published in the *Bulletin of Symbolic Logic* in the near future.

Thank you!