Initial segment complexity and randomness for computable measures

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Joint work with Rupert Hölzl and Wolfgang Merkle

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Introduction

According to the Levin-Schnorr theorem, a sequence $X \in 2^\omega$ is Martin-Löf random if and only if X has sufficiently high initial segment complexity.

The goal of the talk today is to discuss some related results on the growth rate of the initial segment complexity of sequences that are random with respect to a computable measure on 2^ω.

I will focus (for the most part) on computable, *continuous* measures.

Computable measures on 2^ω

For $\sigma \in 2^{<\omega}$, let $[\![\sigma]\!] = \{X \in 2^\omega : \sigma \prec X\}$.

Definition

A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu([\![\sigma]\!])$ is computable as a real-valued function.

In other words, μ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that

$$|\mu([\![\sigma]\!]) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

From now on we will write $\mu(\sigma)$ instead of $\mu([\![\sigma]\!])$.

Martin-Löf randomness with respect to a computable measure

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ *passes* a μ-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$.
- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes every μ-Martin-Löf test.

We will say that X is *proper* if $X \in \mathrm{MLR}_\mu$ for some computable measure μ on 2^ω.

Kolmogorov complexity

Let $U : 2^{<\omega} \to 2^{<\omega}$ be a universal, prefix-free Turing machine.

For each $\sigma \in 2^{<\omega}$, the *prefix-free Kolmogorov complexity* of σ is defined to be

$$K(\sigma) := \min\{|\tau| : U(\tau)\downarrow = \sigma\}$$

The Levin-Schnorr Theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ is Martin-Löf random if and only if

$$\forall n \; K(X \upharpoonright n) \geq n - O(1).$$

More generally, we have the following:

Theorem

Let μ be a computable measure. $X \in 2^\omega$ is μ-Martin-Löf random if and only if

$$\forall n \; K(X \upharpoonright n) \geq -\log(\mu(X \upharpoonright n)) - O(1).$$

Atomic Measures and Continuous Measures

A measure μ on 2^ω is *atomic* if there is some $A \in 2^\omega$ such that $\mu(\{A\}) > 0$.

A is called an *atom* of μ.

For an atomic measure μ, let Atoms_μ be the collection of atoms of μ.

If μ is not atomic, then μ is *continuous*.

A few facts:

- If A is the atom of a computable measure, then $A \in \mathrm{MLR}_\mu$.
- If A is the atom of a computable measure, then A is computable.

A priori complexity

Definition

- A *semi-measure* is a function $\rho : 2^{<\omega} \to [0, 1]$ satisfying
  (i) $\rho(\epsilon) = 1$ and
  (ii) $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.
- A semi-measure ρ is *left-c.e.* if ρ is computably approximable from below.

Fact: There exists a *universal* left-c.e. semi-measure M.

We define the *a priori complexity* of $\sigma \in 2^{<\omega}$ to be

$$KA(\sigma) := -\log M(\sigma).$$

Complex and strongly complex sequences

Recall that an order function $h : \omega \to \omega$ is an unbounded, non-decreasing function.

Definition

Let $X \in 2^\omega$.

- X is *complex* if there is a computable order function $h : \omega \to \omega$ such that

$$\forall n \; K(X \upharpoonright n) \geq h(n).$$

- X is *strongly complex* if there is a computable order function $g : \omega \to \omega$ such that

$$\forall n \; KA(X \upharpoonright n) \geq g(n).$$

Proposition

X is complex if and only if X is strongly complex.

Our main question

Given that proper sequences must have sufficiently high initial segment complexity, it is reasonable to ask:

What is the relationship between the collection of proper sequences and the collection of complex sequences?

- Not every proper sequence is complex.
- Not every complex sequence is proper.

A sufficient condition for complexity

Theorem (Hölzl, Merkle, Porter)

If $X \in 2^\omega$ is Martin-Löf random with respect to a computable, continuous measure μ, then X is complex.

This follows from the following two results.

Lemma

Let μ be a computable, continuous measure and let $X \in \mathsf{MLR}_\mu$. Then there is some Martin-Löf random $Y \leq_{\mathrm{tt}} X$.

Lemma

If X is complex and $X \leq_{\mathrm{wtt}} Y$, then Y is complex.

What about the converse?

As stated earlier, the converse of the previous theorem doesn't hold: there are complex sequences that are not proper.

However, we do have a partial converse.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathsf{MLR}_\mu$ for some computable, continuous measure μ.

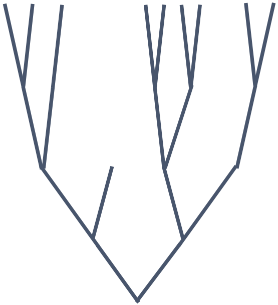
A useful lemma

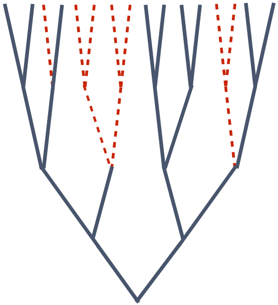
Lemma

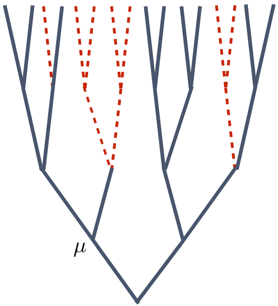
Suppose that

- μ *is a computable measure,*
- $X \in \mathsf{MLR}_\mu$ *is non-computable,*
- $\mathcal{P}$ *is a* Π^0_1 *class with no computable members, and*
- $X \in \mathcal{P}$.

Then there is some computable, continuous measure ν *such that* $X \in \mathsf{MLR}_\nu$.



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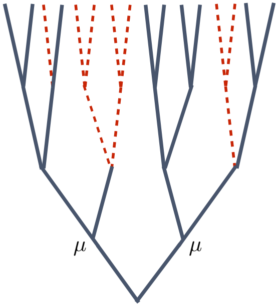
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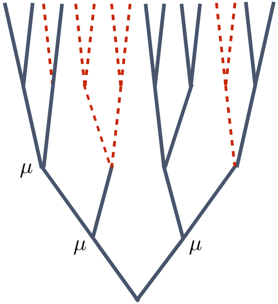

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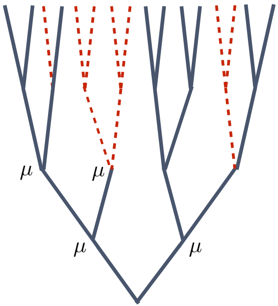
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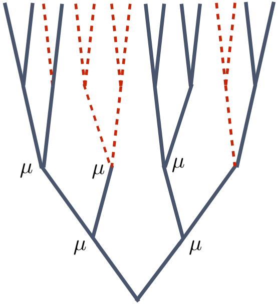
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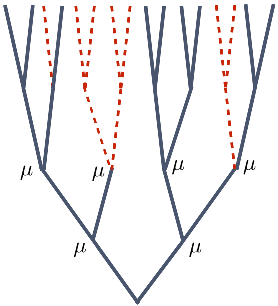

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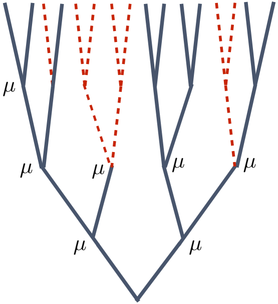

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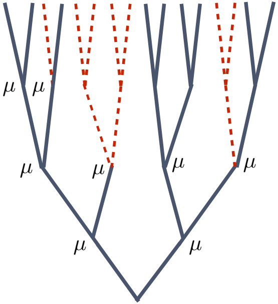
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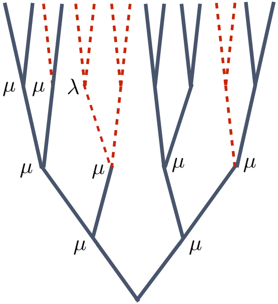

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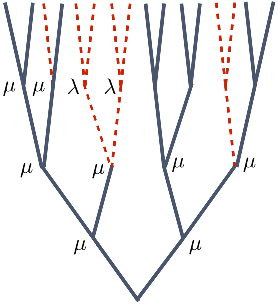
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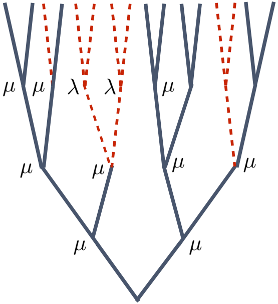

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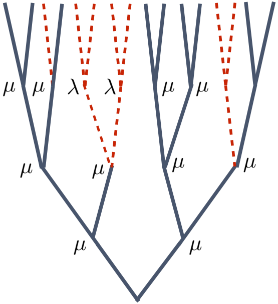

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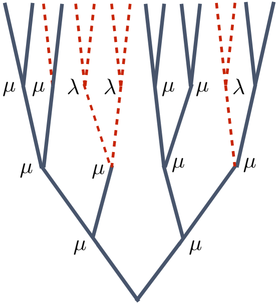

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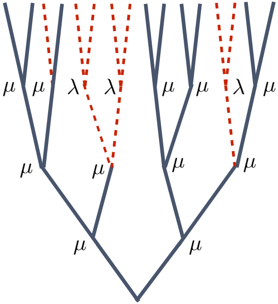
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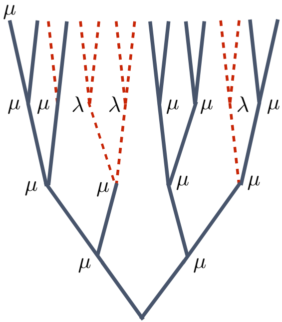
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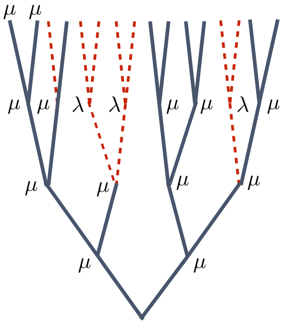

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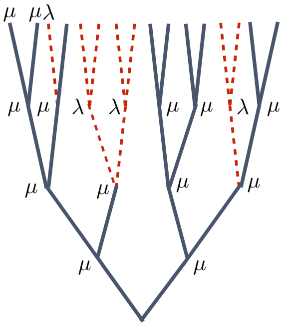

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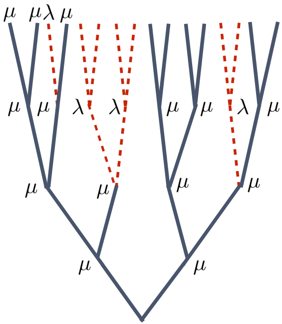

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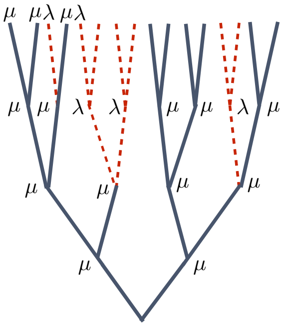

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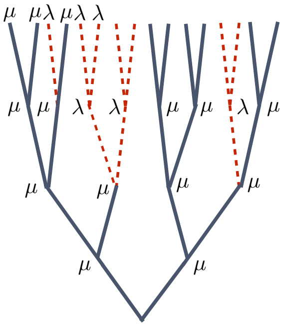
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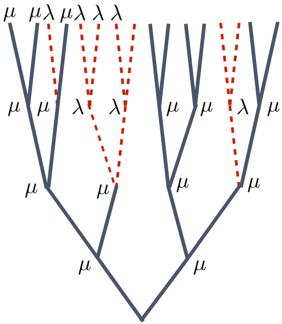

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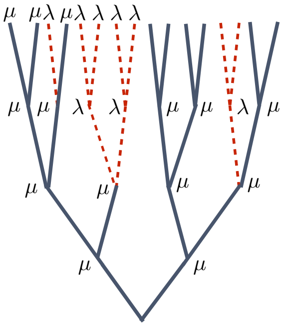
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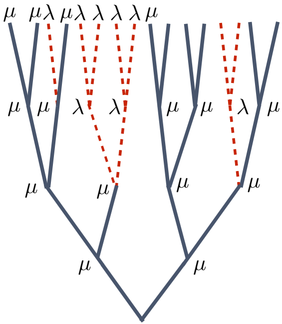
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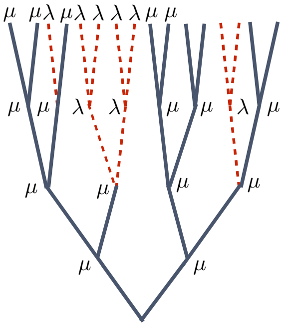

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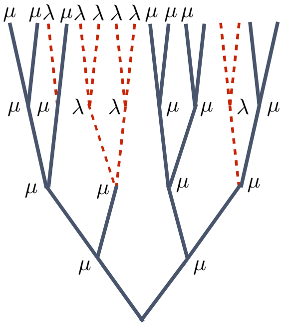
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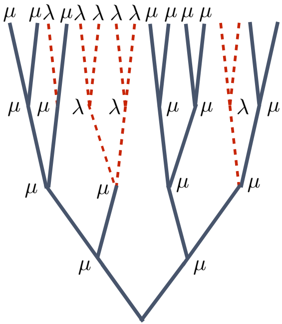
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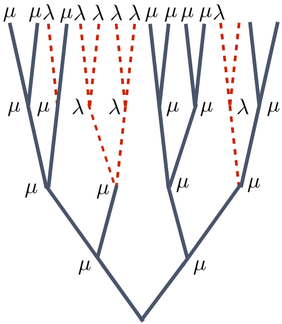
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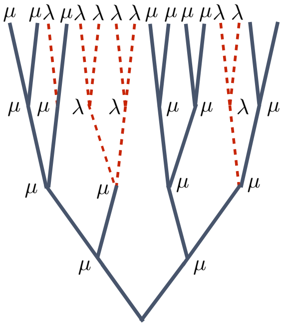
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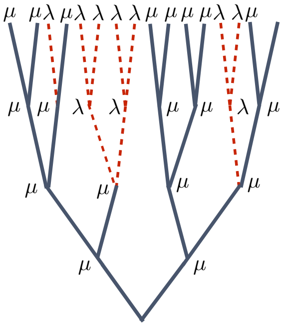
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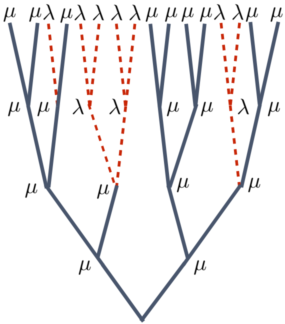
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Establishing the partial converse

Theorem

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

To prove this theorem, let h be the computable order function that witnesses that X is complex.

Then we apply the previous lemma to the Π^0_1 class

$$\{A \in 2^\omega : K(A{\upharpoonright}n) \geq h(n)\}.$$

Connection to semigenericity

Definition

$X \in 2^\omega$ is *semigeneric* if for every Π^0_1 class $\mathcal{P}$ with $X \in \mathcal{P}$, $\mathcal{P}$ contains some computable member.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. *$X \in \mathrm{MLR}_\mu$ for some computable, continuous μ.*
2. *X is complex.*
3. *X is not semigeneric.*
4. *X is hyperavoidable.*
5. *X is avoidable.*

A follow-up result

Definition

Let μ be a continuous measure. Then the *granularity function of μ*, denoted g_μ, is the order function mapping n to the least ℓ such that $\mu(\sigma) < 2^{-n}$ for every σ of length ℓ.

Theorem (Hölzl, Merkle, Porter)

Let μ be a computable, continuous measure and let $X \in \mathrm{MLR}_\mu$. Then we have

$$\forall n \; KA(X \upharpoonright n) \geq g_\mu^{-1}(n) - O(1).$$

- There is a computable, continuous measure μ such that the granularity function g_μ of μ is not computable.
- For every computable, continuous measure μ, there is a computable order function $f : \omega \to \omega$ such that
$$|f(n) - g_\mu(n)^{-1}| \leq O(1).$$

A question about uniformity

Question

If we have a computable, atomic measure μ such that

$$\forall X \in 2^\omega \ (X \in \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \ \Rightarrow \ X \text{ is complex}),$$

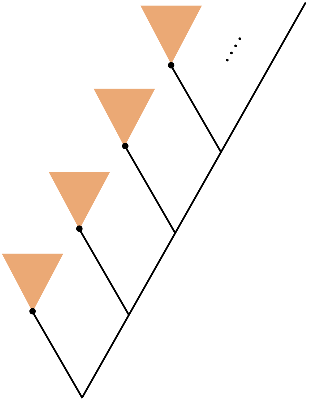
is there a computable, continuous measure ν such that

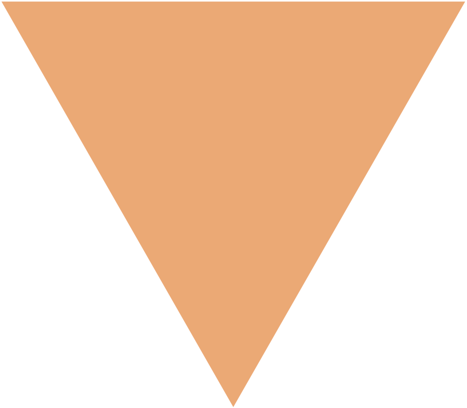
$$\mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \subseteq \mathrm{MLR}_\nu?$$

Theorem (Hölzl, Merkle, Porter)

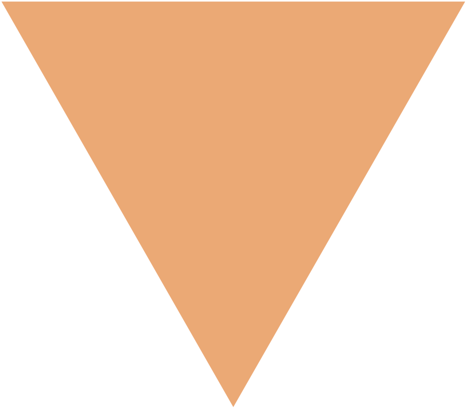
There is a computable, atomic measure μ *such that*

- *every* $X \in \mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu$ *is complex but*
- *there is no computable, continuous measure* ν *such that* $\mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \subseteq \mathsf{MLR}_\nu$.



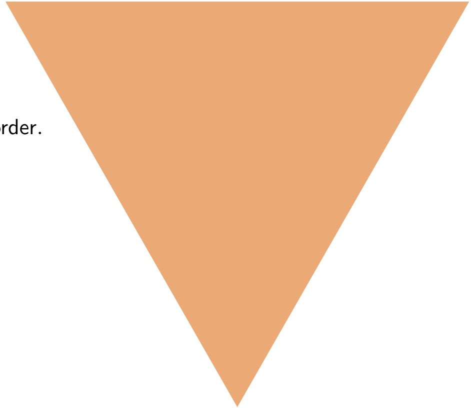


the i^{th} neighborhood



the i^{th} neighborhood

Suppose that ϕ_i is an order.



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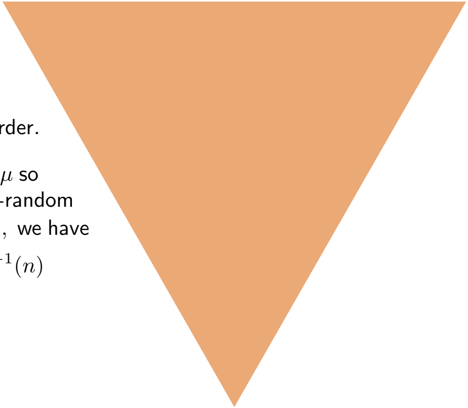

Suppose that ϕ_i is an order.

We define the measure μ so that for any complex μ-random X in this neighborhood, we have

$$KA(X\restriction n) < \phi_i^{-1}(n)$$

for almost every n.

the i^{th} neighborhood

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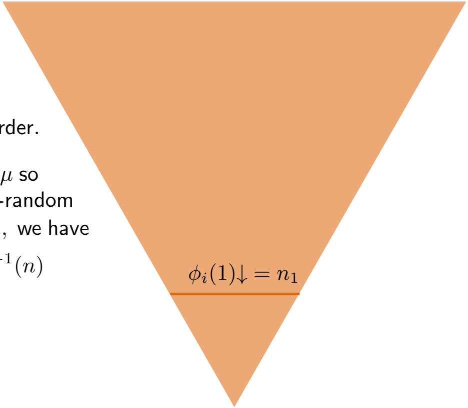

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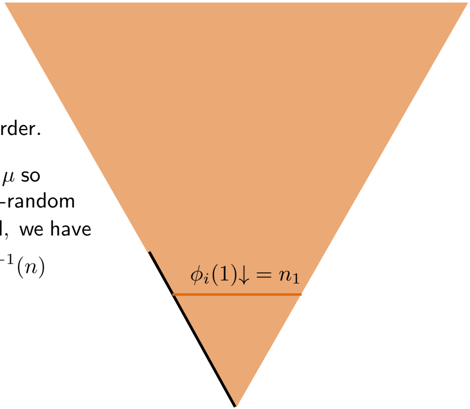

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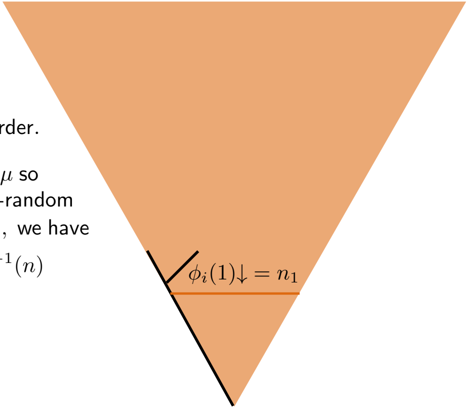

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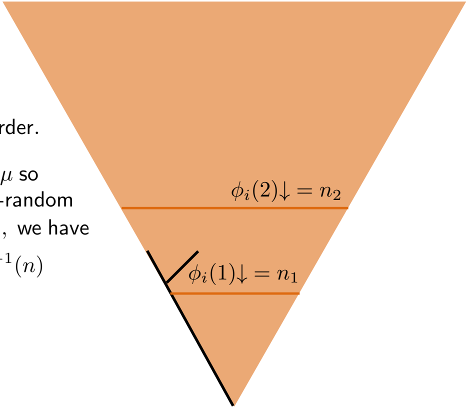
the i^{th} neighborhood



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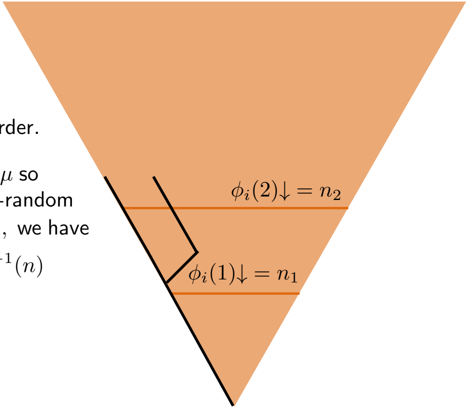

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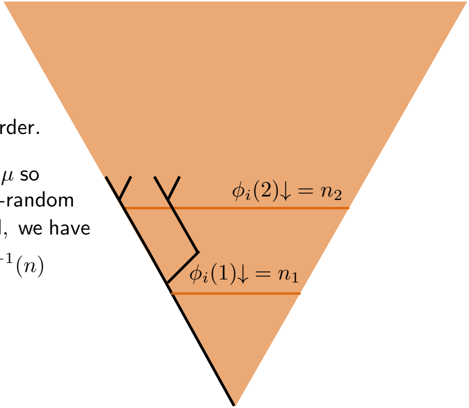
the i^{th} neighborhood



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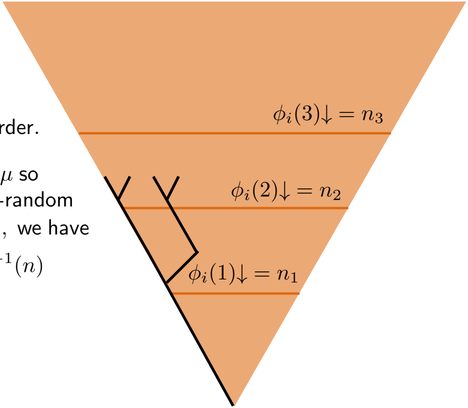

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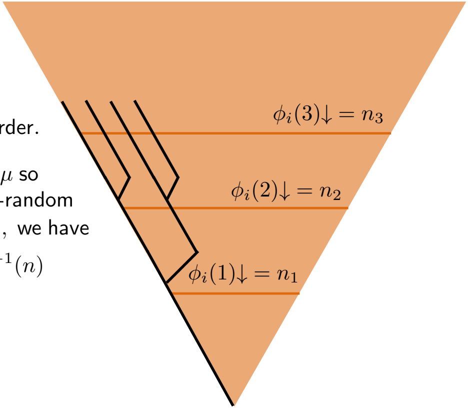

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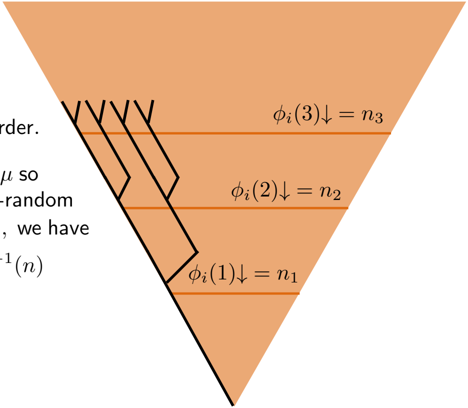

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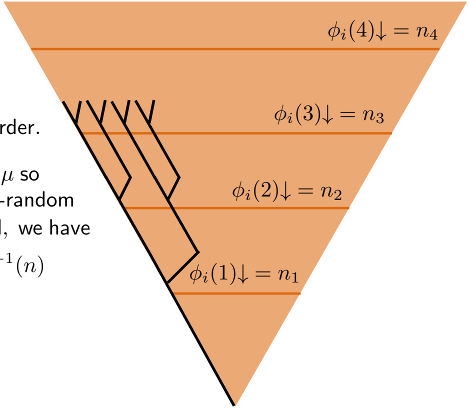

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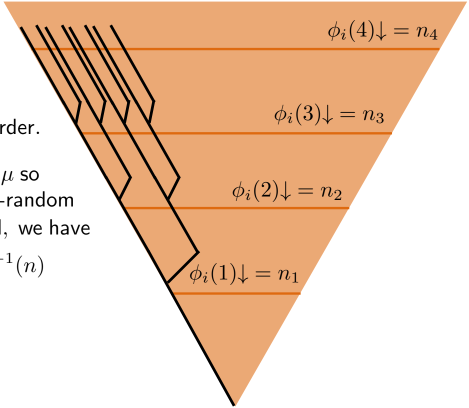

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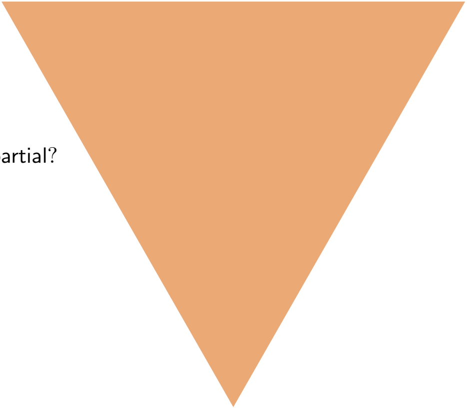
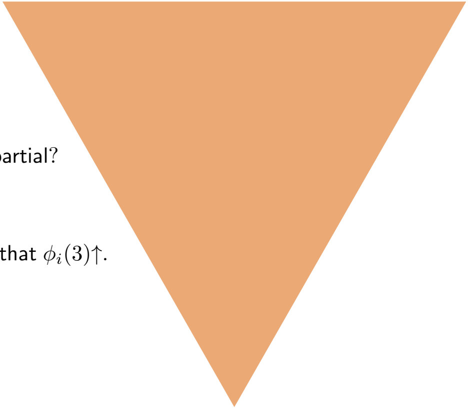


What happens if ϕ_i is partial?

the i^{th} neighborhood



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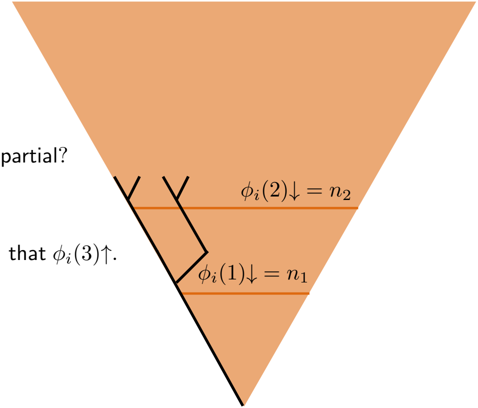

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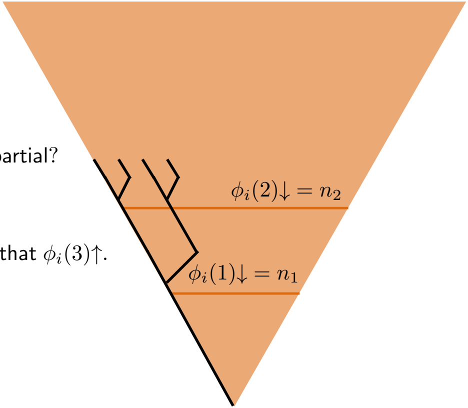

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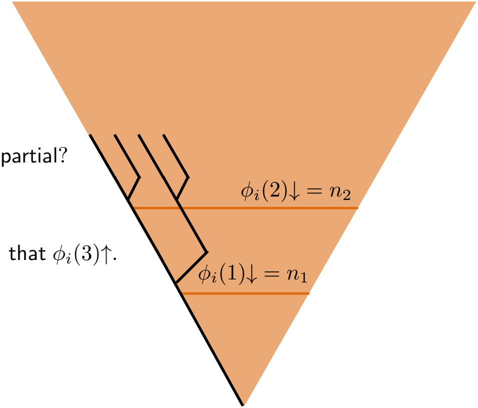

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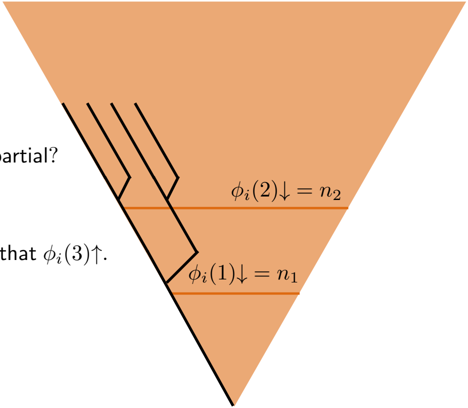

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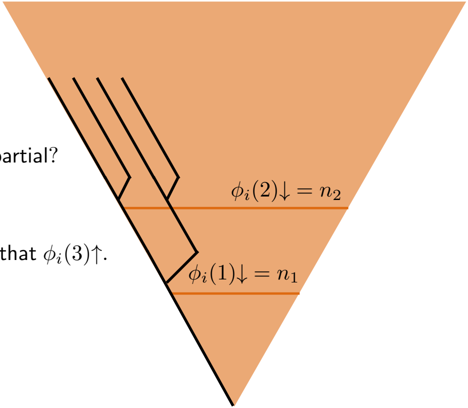

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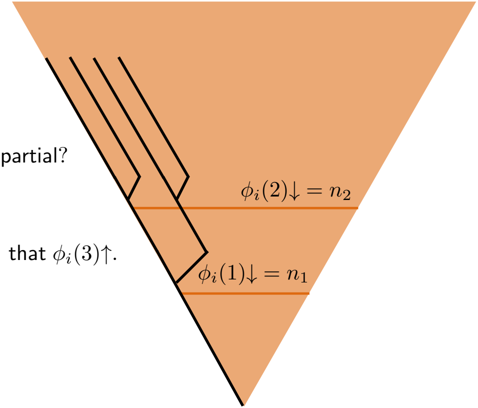

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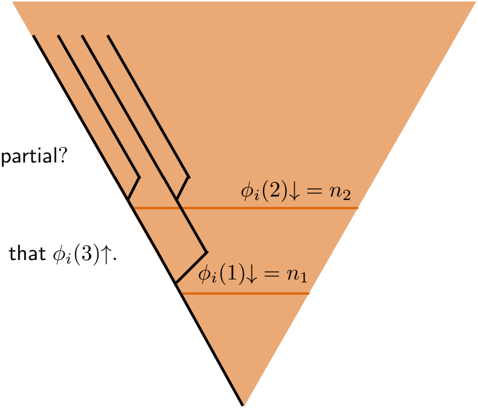

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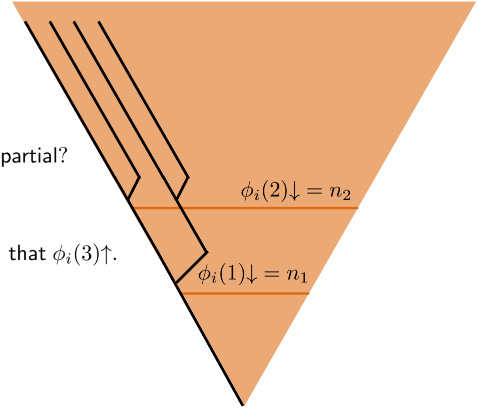

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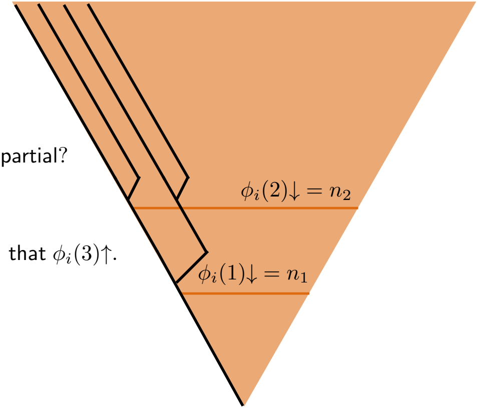

Let $\llbracket \sigma_i \rrbracket$ be the i^{th} neighborhood.

One can verify that

- if ϕ_i is partial, then $\llbracket \sigma_i \rrbracket \cap \mathrm{MLR}_\mu \subseteq \mathrm{Atoms}_\mu$;
- if ϕ_i is total, then $\llbracket \sigma_i \rrbracket \cap \mathrm{Atoms}_\mu = \emptyset$ and every $X \in \mathrm{MLR}_\mu \cap \llbracket \sigma_i \rrbracket$ is complex.

Lastly, if there is some computable, continuous ν such that $\mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \subseteq \mathrm{MLR}_\nu$, then there is a computable order $f = \phi_i$ such that for every $X \in \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu$,

$$KA(X{\upharpoonright}n) \geq f^{-1}(n) - O(1)$$

for every n, which yields a contradiction.

Thank you!