Strong Difference Randomness

Christopher Porter
LIAFA
Université Paris Diderot - Paris 7

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(Joint work with Laurent Bienvenu)

Introduction, I

Consider the following two theorems:

Theorem (Nies, Stephan, Terwijn): Every 2-random forms a minimal pair with every 2-generic.

Theorem (Kautz): Every 2-random has hyperimmune degree.

Introduction, II

In the course of studying the proofs of these two theorems, we came to realize that these results are not optimal.

However, the notion of randomness in terms of which we improved these results had not been previously isolated and studied.

Today I will introduce this notion, which we refer to as *strong difference randomness*.

Another definition of randomness?

My goal is to convince you that strong difference randomness is a reasonable definition of randomness to study:

- SDR is useful for studying "almost all" properties of the Turing degrees.

- SDR has a natural connection to certain probabilistic algorithms.

Difference randomness, I

To understand strong difference randomness, we should first discuss difference randomness.

Standard "test" definitions of algorithmic randomness:

- Defined in terms of a uniform collection of effectively open classes.

Difference randomness:

- Defined in terms of a uniform collection of *differences* of effectively open classes.

Difference randomness, II

A *difference test* is a uniform collection of pairs of effectively open classes $\{(\mathcal{U}_n, \mathcal{V}_n)\}_{n\in\omega}$ such that

$$\lambda(\mathcal{U}_n \setminus \mathcal{V}_n) \leq 2^{-n}$$

for every $n \in \omega$.

$X \in 2^\omega$ *passes the difference test* $\{(\mathcal{U}_n, \mathcal{V}_n)\}_{n\in\omega}$ if

$$X \notin \bigcap_{i\in\omega} (\mathcal{U}_i \setminus \mathcal{V}_i).$$

$X \in 2^\omega$ is *difference random* if X passes every difference test.

Difference randomness, III

Difference randomness has proven to be a very useful notion of randomness.

Franklin and Ng have shown that difference random reals are precisely the Turing incomplete Martin-Löf random reals.

By a result of Stephan, difference random reals are precisely the Martin-Löf random reals that do not compute any complete, consistent extension of PA.

Two approaches to tests

Let $\{\mathcal{S}_n\}_{n\in\omega}$ be some collection of sets in terms of which one defines a "test" for randomness.

There are two ways for a sequence $X \in 2^\omega$ to pass the test $\{\mathcal{S}_n\}_{n\in\omega}$:

- $X \notin \bigcap_{i\in\omega} \mathcal{S}_i$
- $X \notin \mathcal{S}_i$ for all but finitely many $i \in \omega$.

The latter condition is sometimes called the "Solovay condition".

The definition of SDR

We obtain the definition of strong difference randomness by considering difference tests with the Solovay condition.

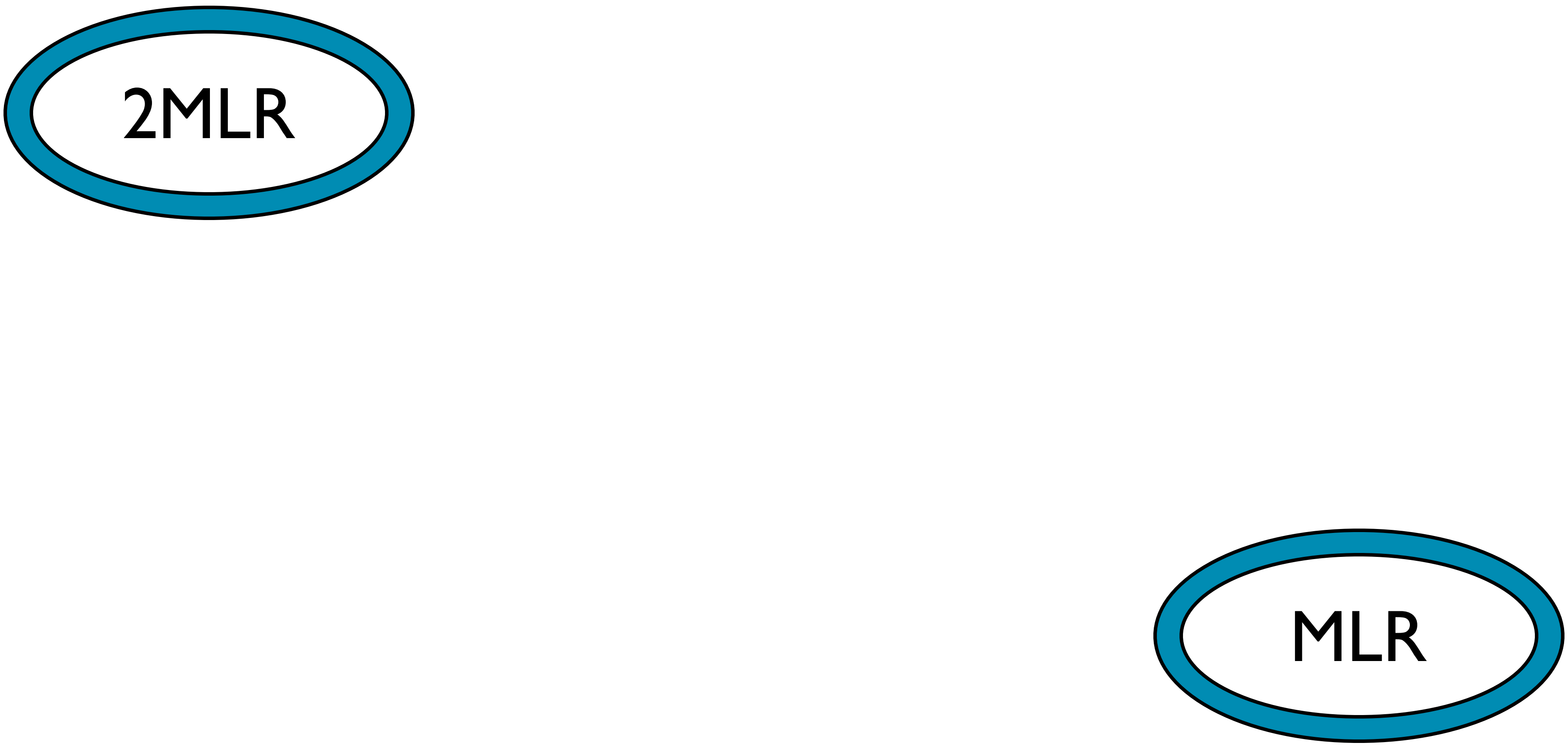
That is, $X \in 2^\omega$ is *strongly difference random* if for every difference test $\{(\mathcal{U}_n, \mathcal{V}_n)\}_{n\in\omega}$, we have

$$X \notin \mathcal{U}_i \setminus \mathcal{V}_i$$

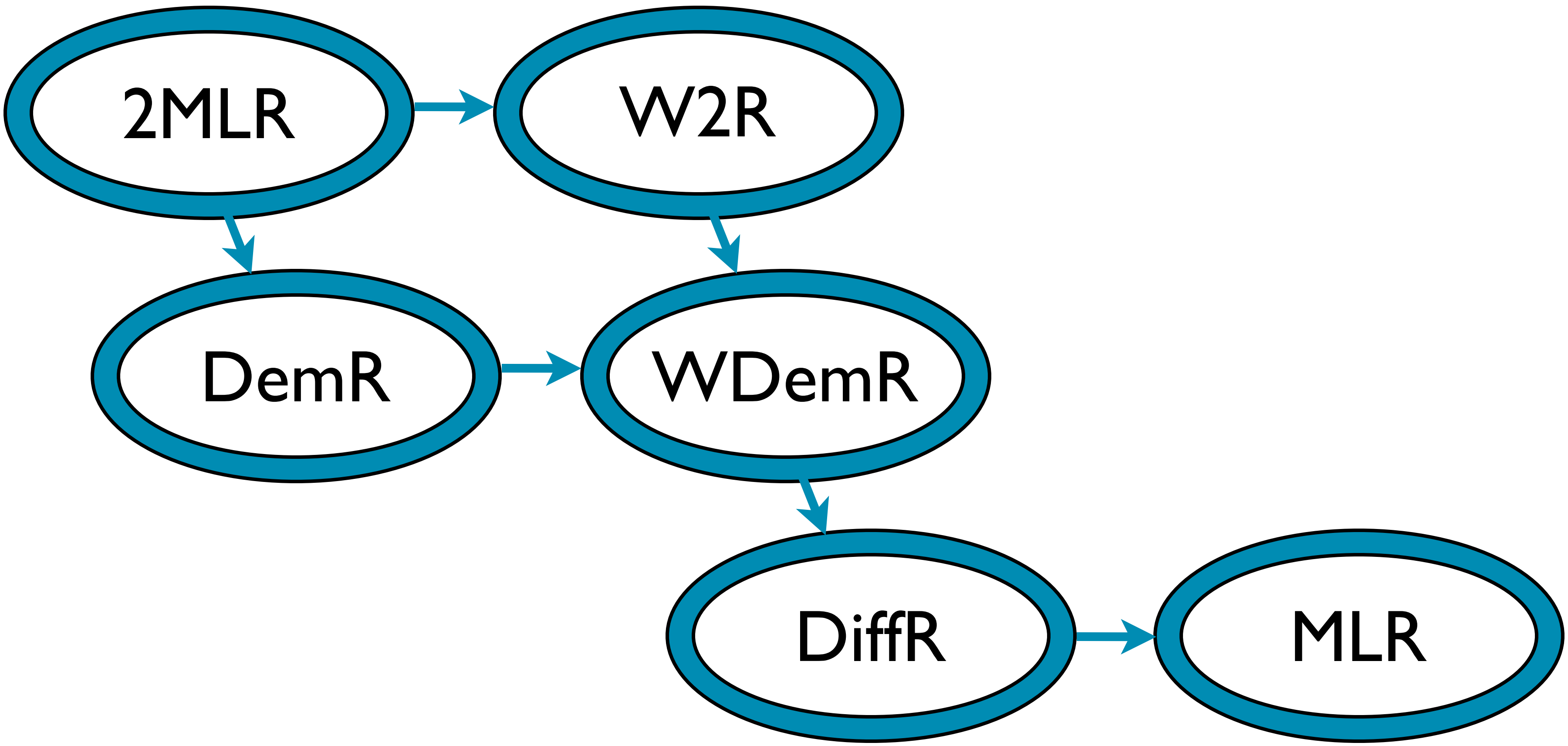
for all but finitely many $i \in \omega$.

SDR vs other notions of randomness

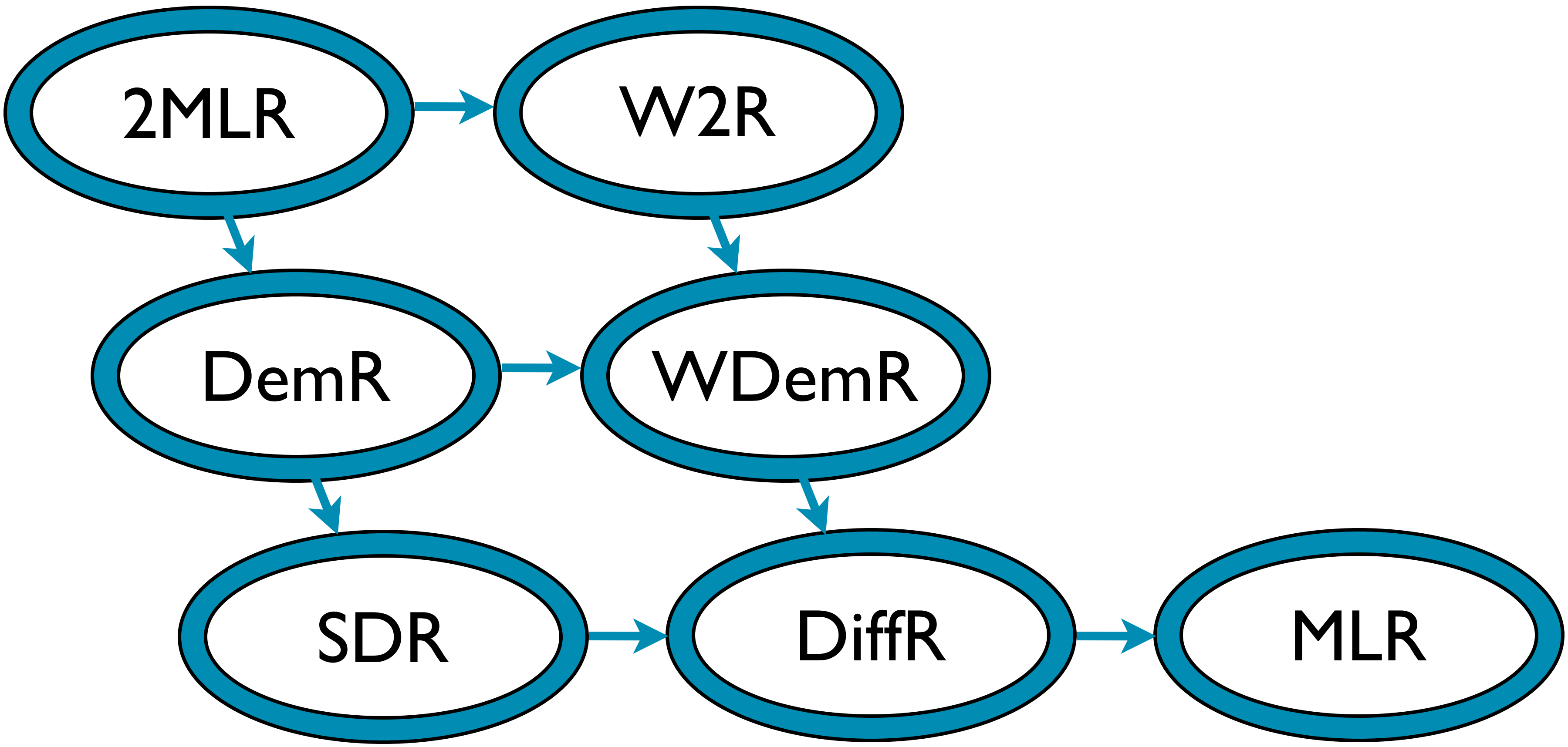
SDR vs other notions of randomness



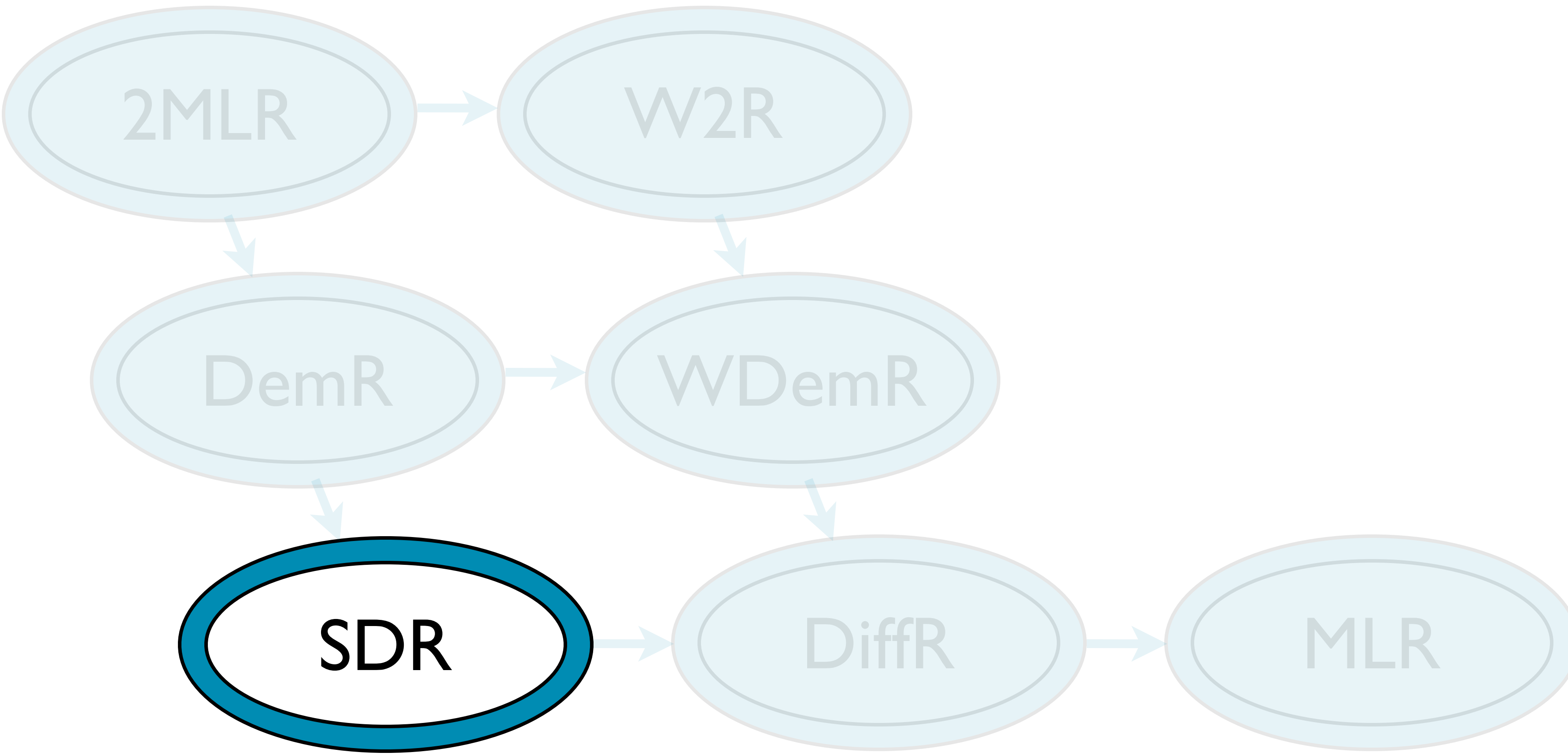
SDR vs other notions of randomness



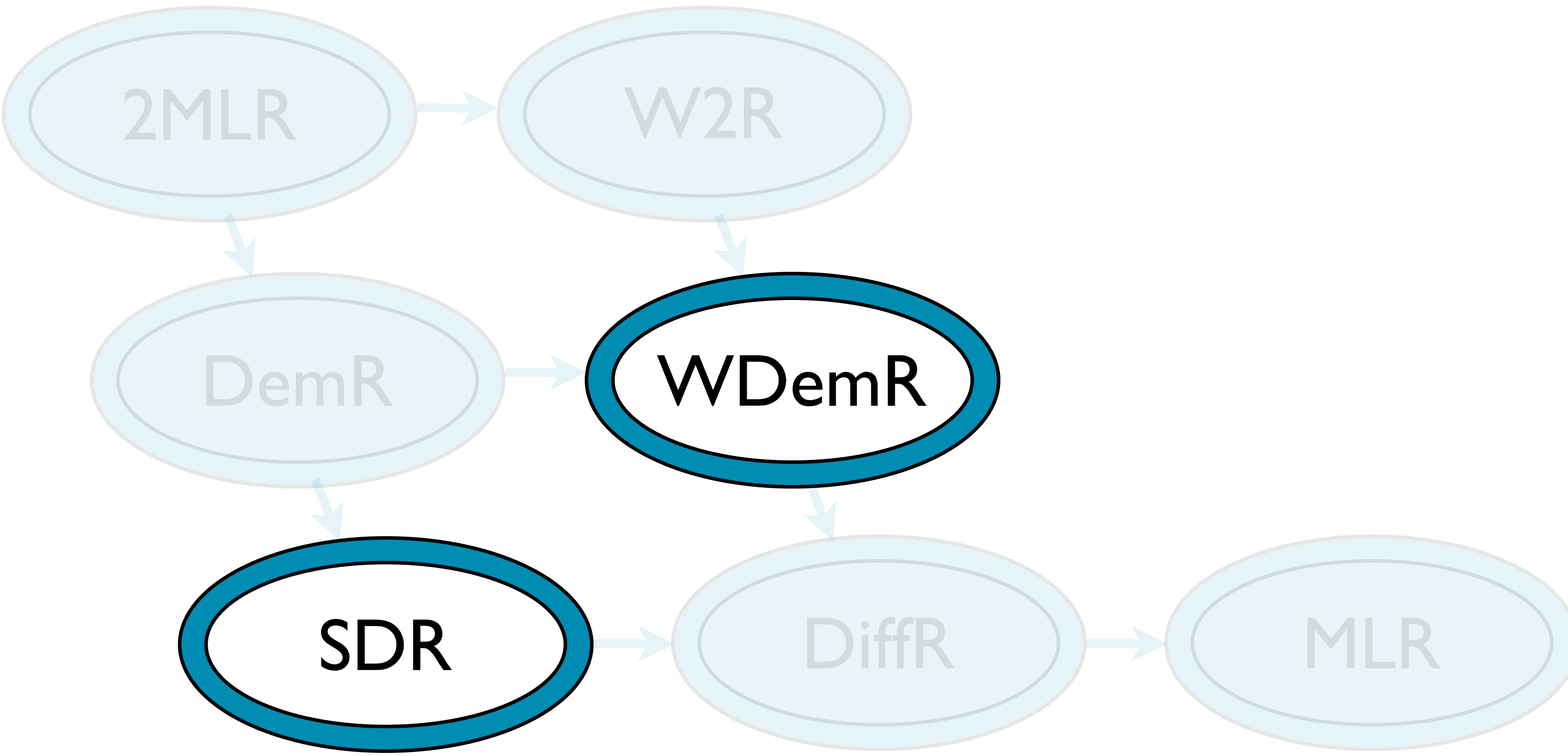
SDR vs other notions of randomness



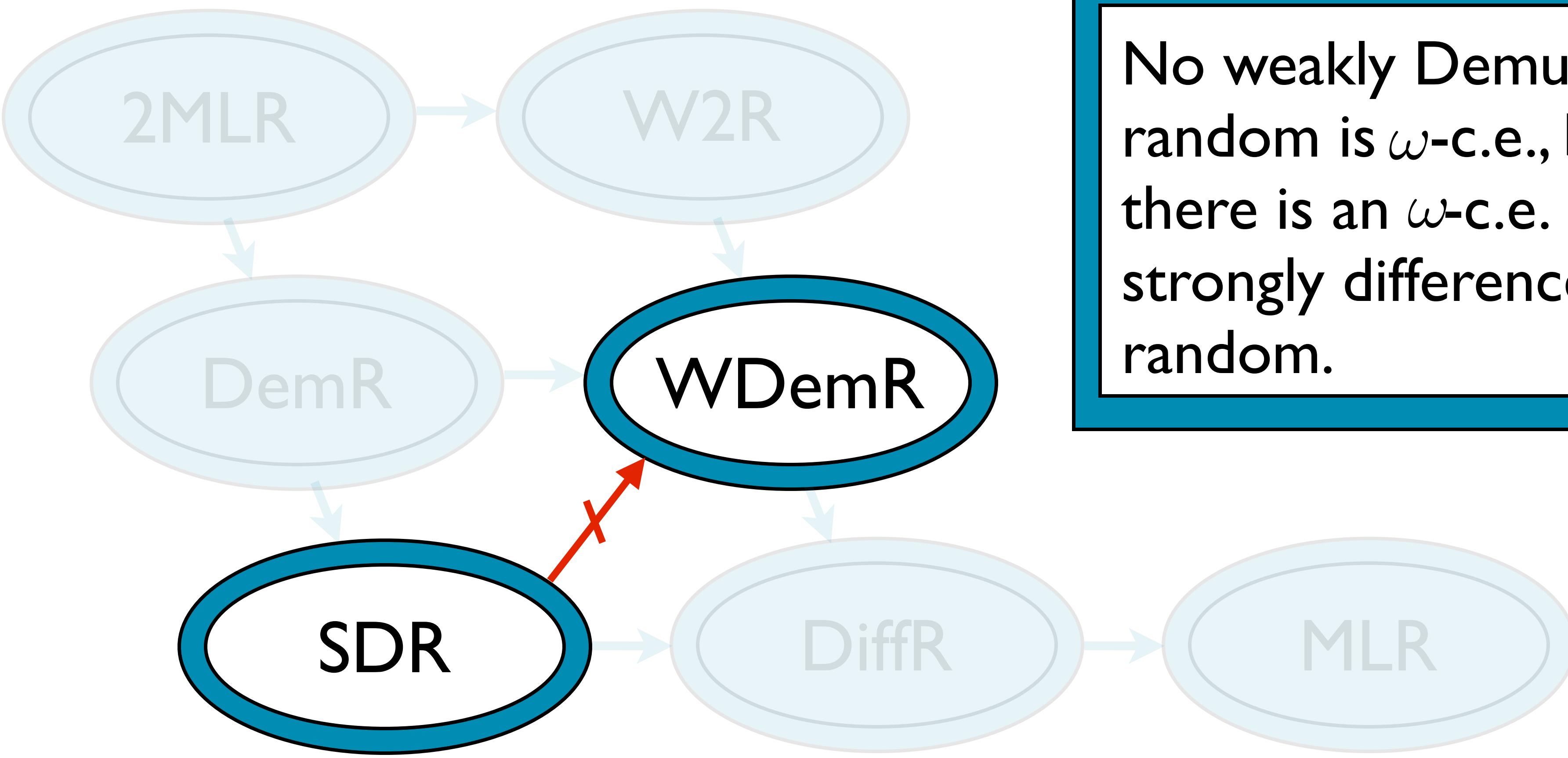
SDR vs ???



SDR vs WDemR, part 1

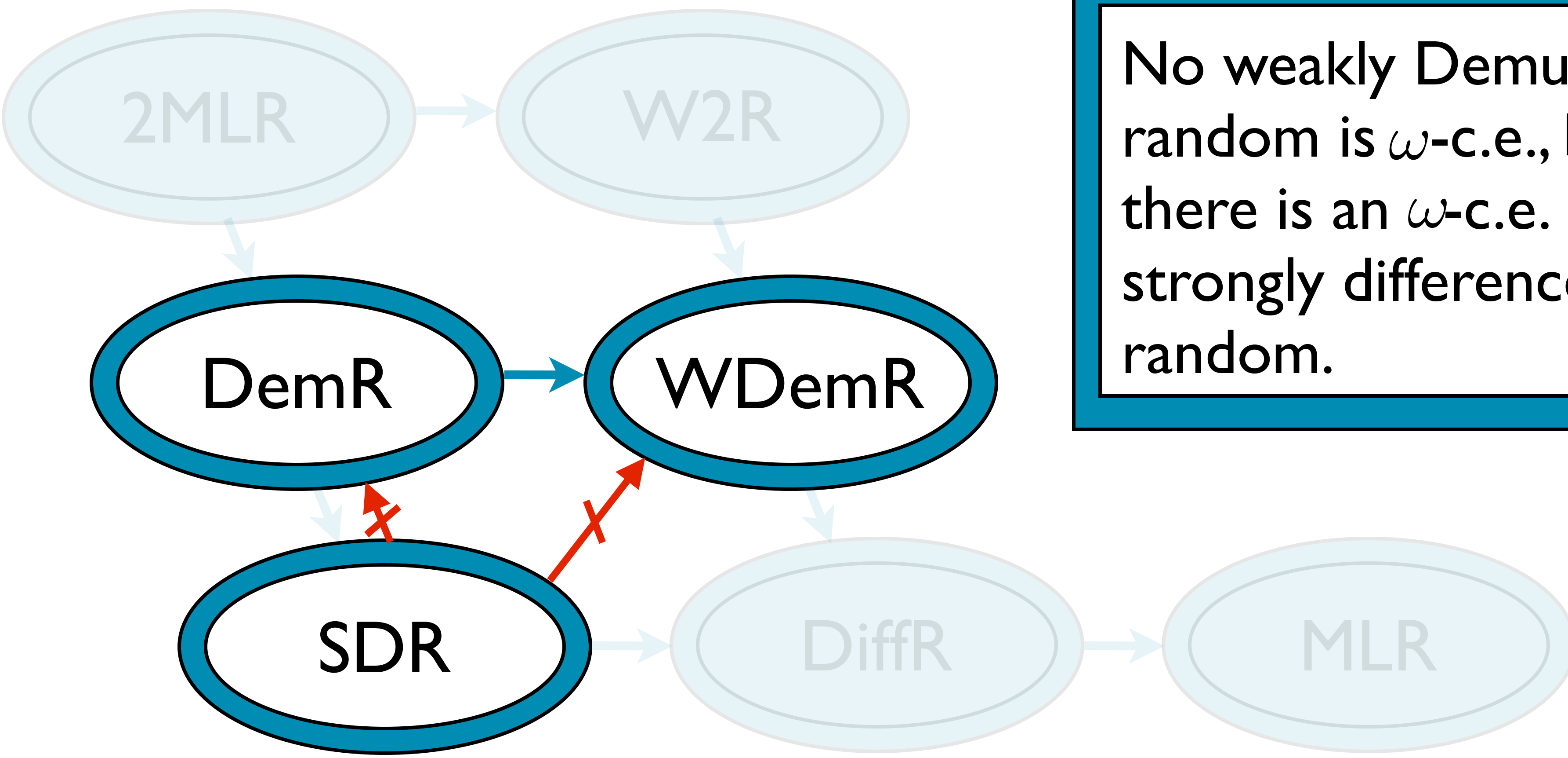


SDR vs WDemR



No weakly Demuth random is ω-c.e., but there is an ω-c.e. strongly difference random.

SDR vs DemR



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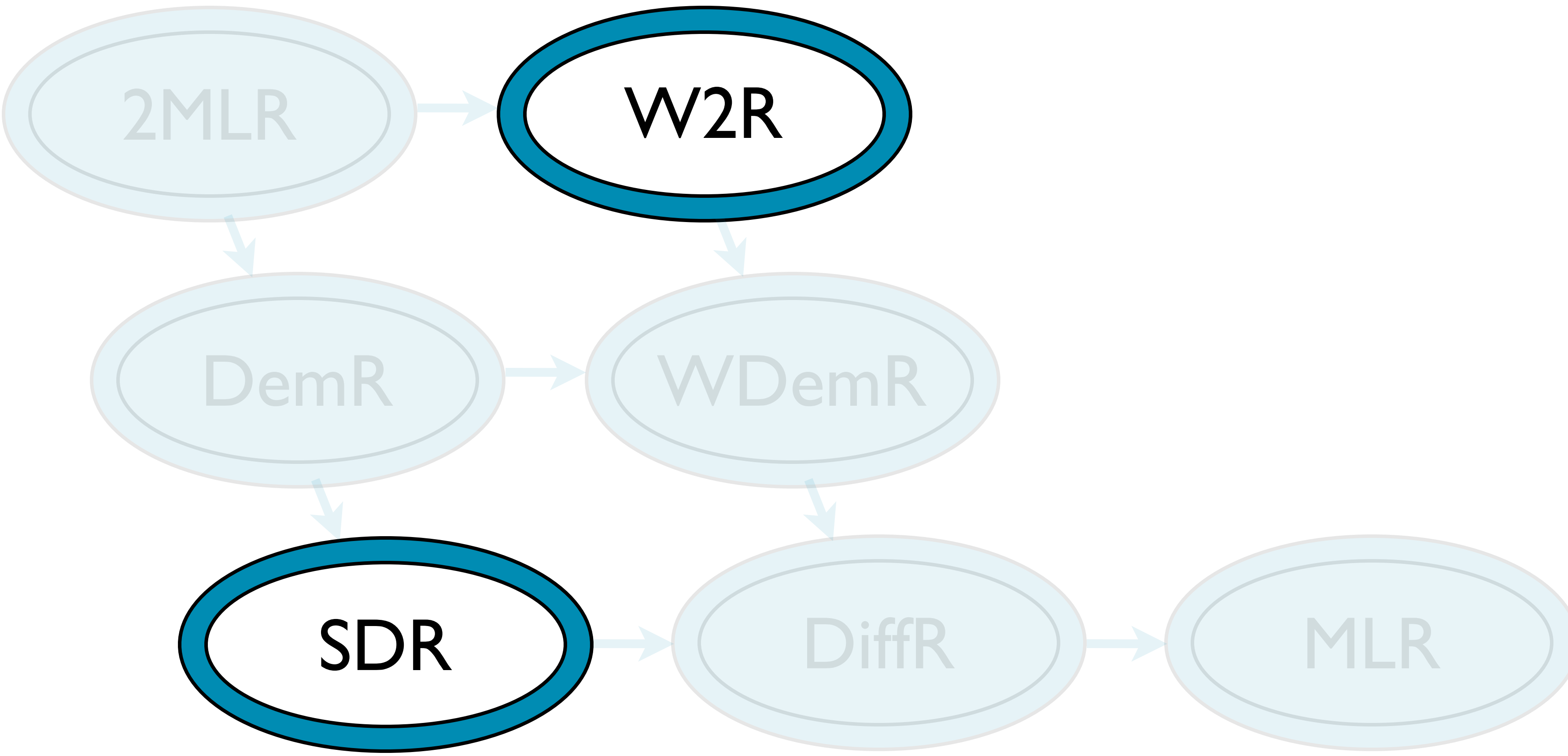
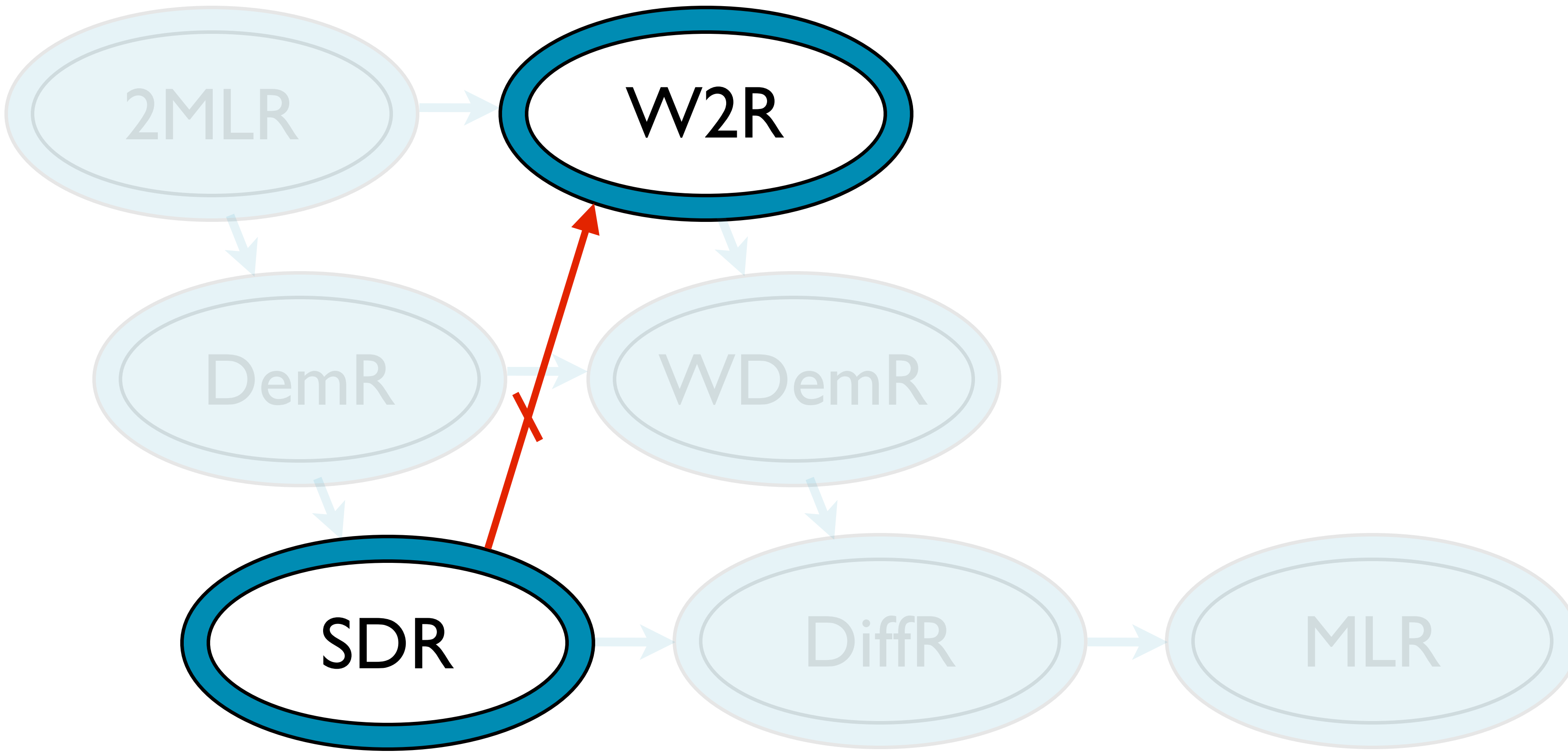
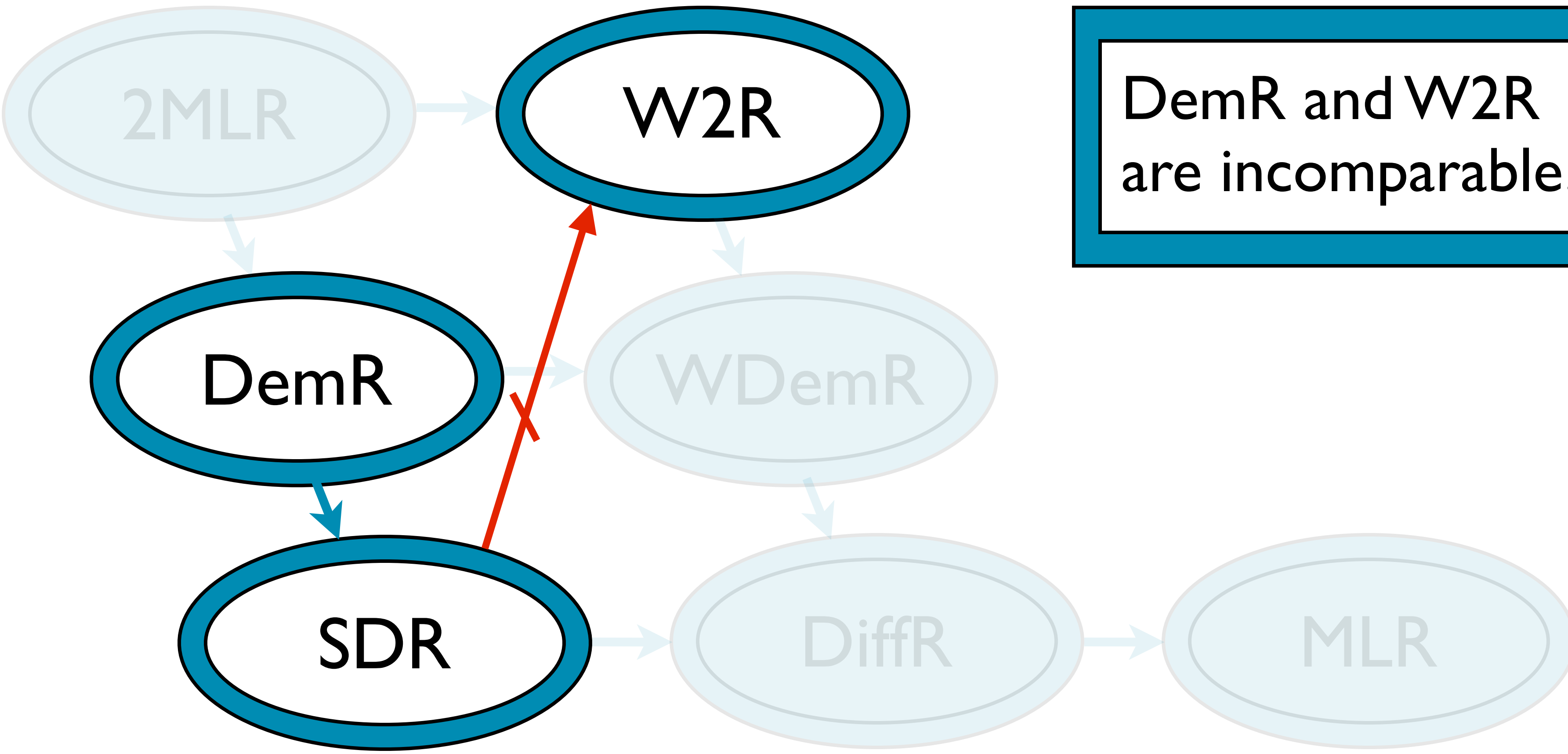
SDR vs W2R



SDR vs W2R



SDR vs W2R



DemR and W2R are incomparable.

The typical Turing degree

Recently, there has been interest in studying the behavior of the typical Turing degree.

For a given property P satisfied by measure one many sequences (and thus measure one many Turing degrees), we ask:

- "What level of randomness is sufficient for P to hold?"
- "What level of randomness is necessary for P to hold?"

Some examples, I

- Almost every Turing degree is generalized low ($X' \equiv_T X \oplus \emptyset'$).
- Almost every Turing degree forms a minimal pair with every 2-generic.
- Almost every Turing degree is hyperimmune.
- Almost every Turing degree computes a 1-generic sequence.

Some examples, II

It has been shown that 2-randomness is sufficient for these properties to hold:

- Every 2-random sequence is GL_1.
- Every 2-random sequence forms a minimal pair with every 2-generic.
- Every 2-random sequence has hyperimmune degree.
- Every 2-random sequence computes a 1-generic sequence.

The main results

Each of the results from the previous slide can be improved:

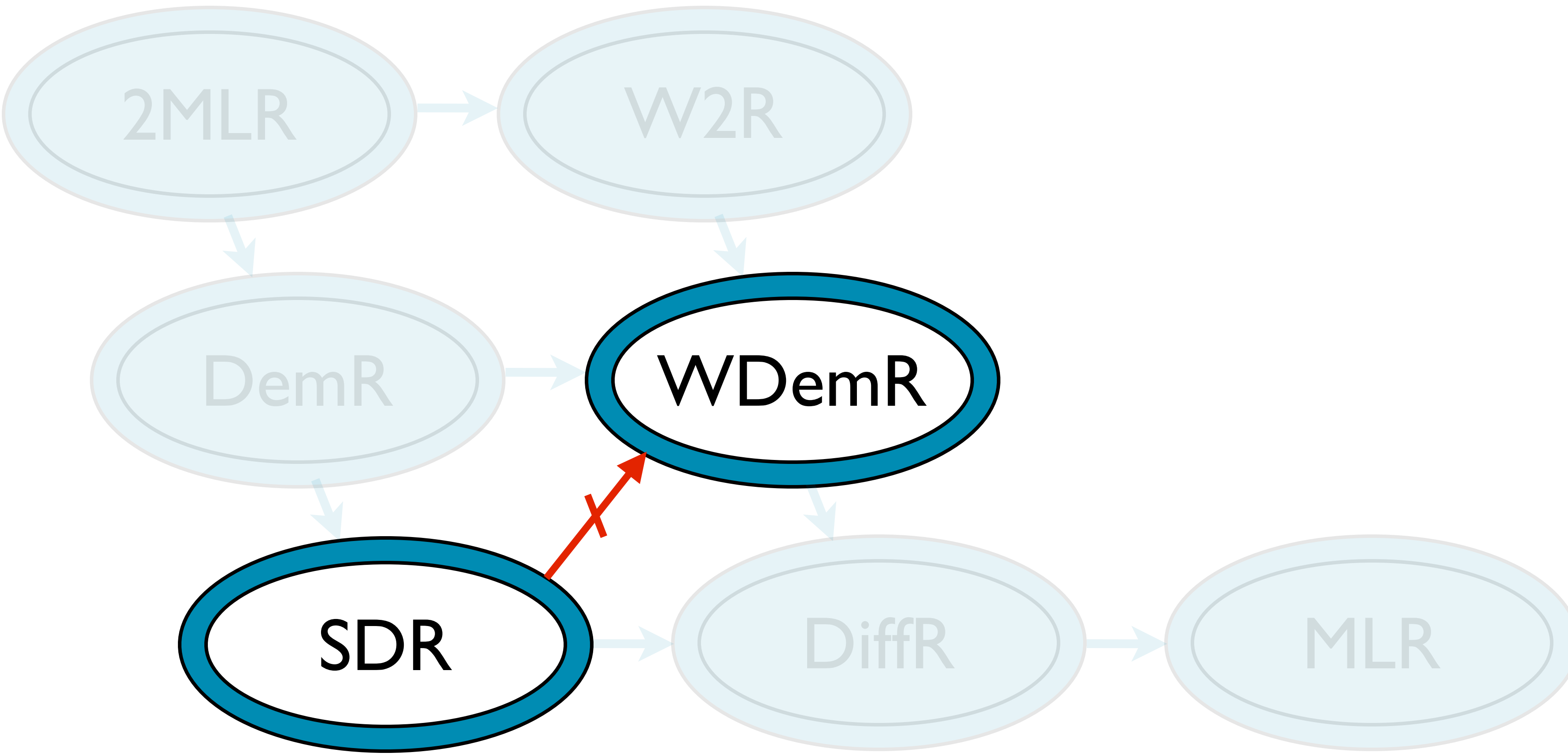
- Every strongly difference random sequence is GL_1.
- Every strongly difference random sequence forms a minimal pair with every 2-generic.
- Every strongly difference random sequence has hyperimmune degree.
- Every strongly difference random sequence computes a 1-generic sequence.

The main results

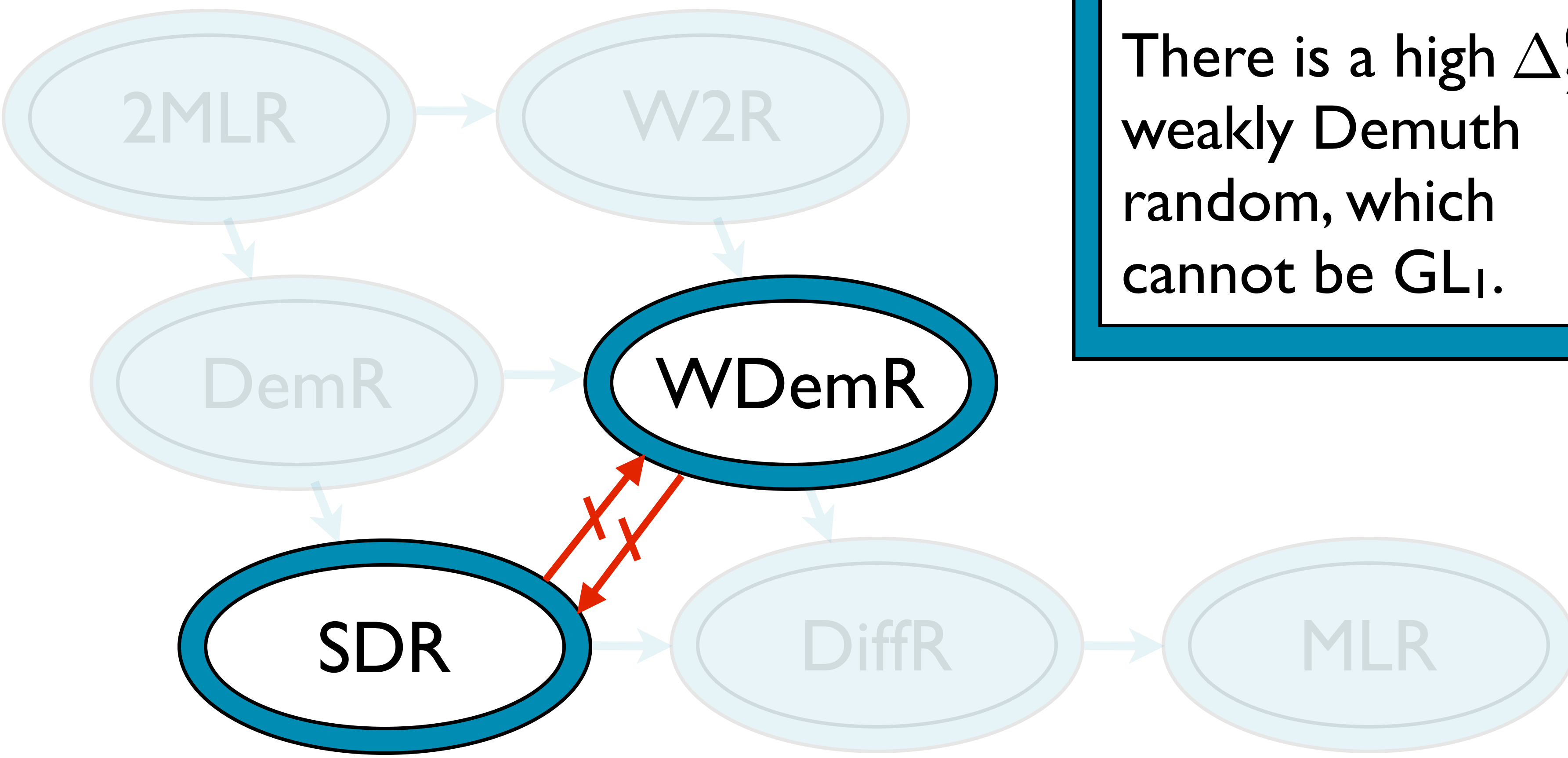
Each of the results from the previous slide can be improved:

- Every strongly difference random sequence is GL_1.
- Every strongly difference random sequence forms a minimal pair with every 2-generic.
- Every strongly difference random sequence has hyperimmune degree.
- Every strongly difference random sequence computes a 1-generic sequence.

SDR vs WDemR

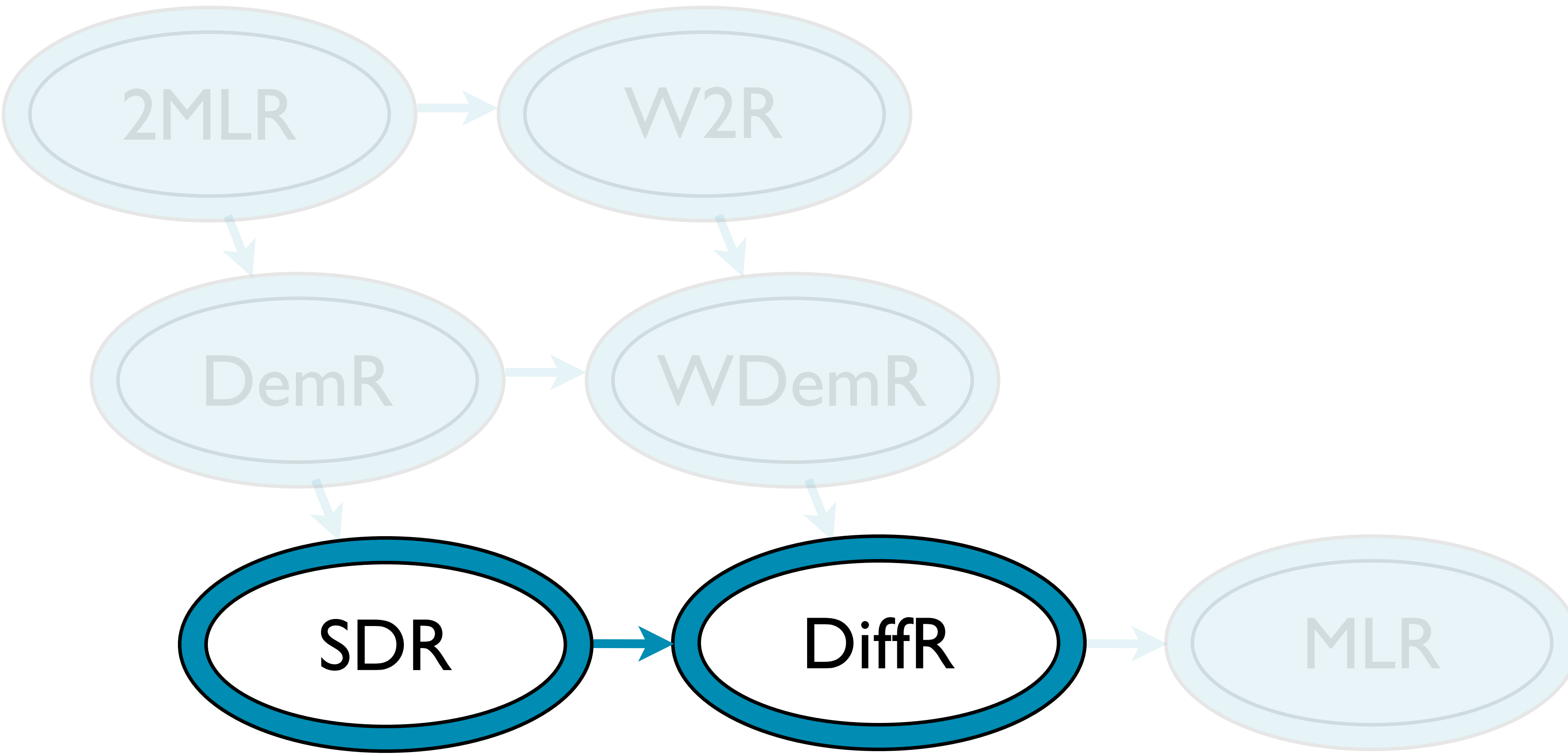


SDR vs WDemR

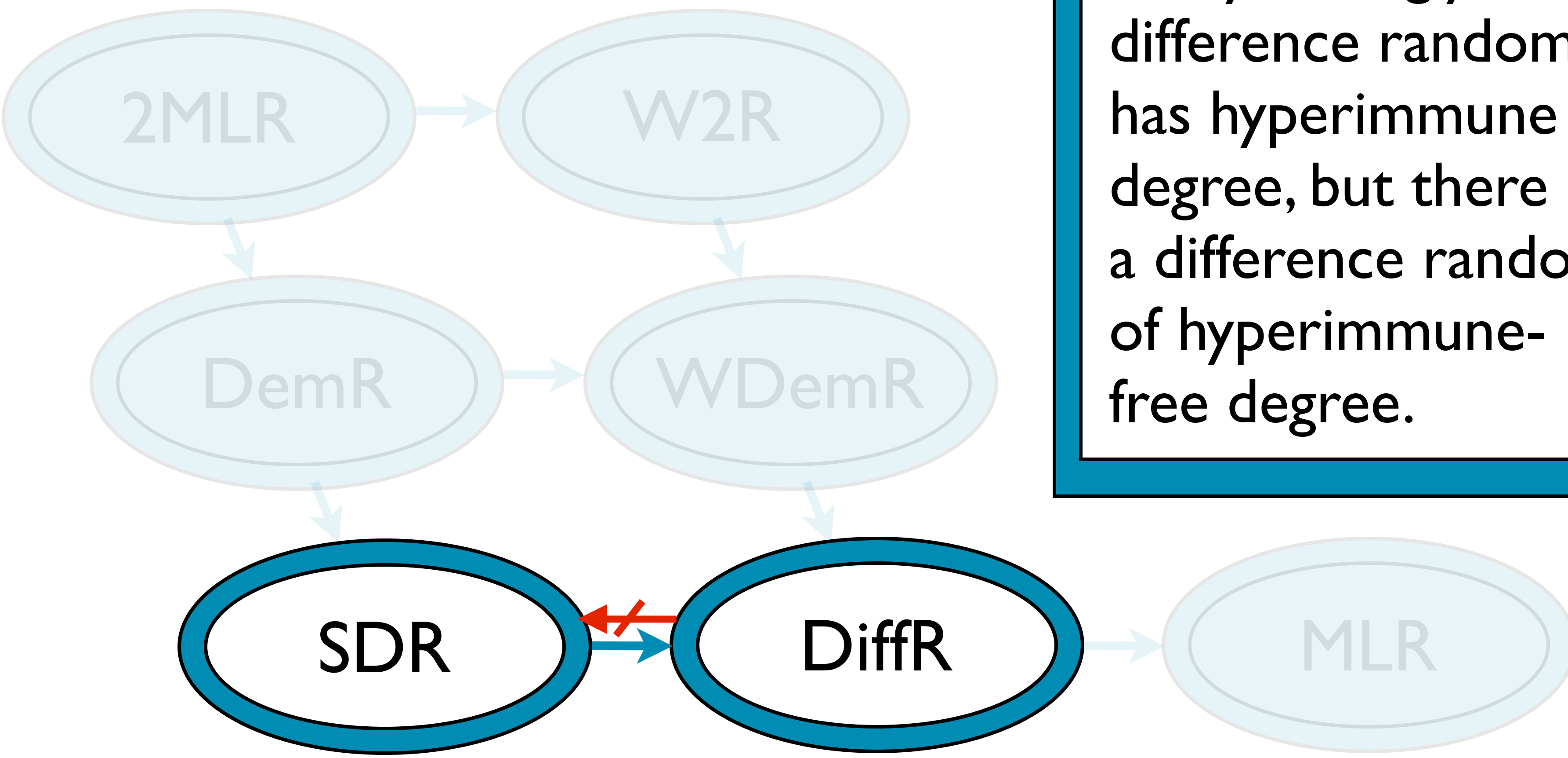


There is a high Δ^0_2 weakly Demuth random, which cannot be GL_1.

SDR vs DiffR

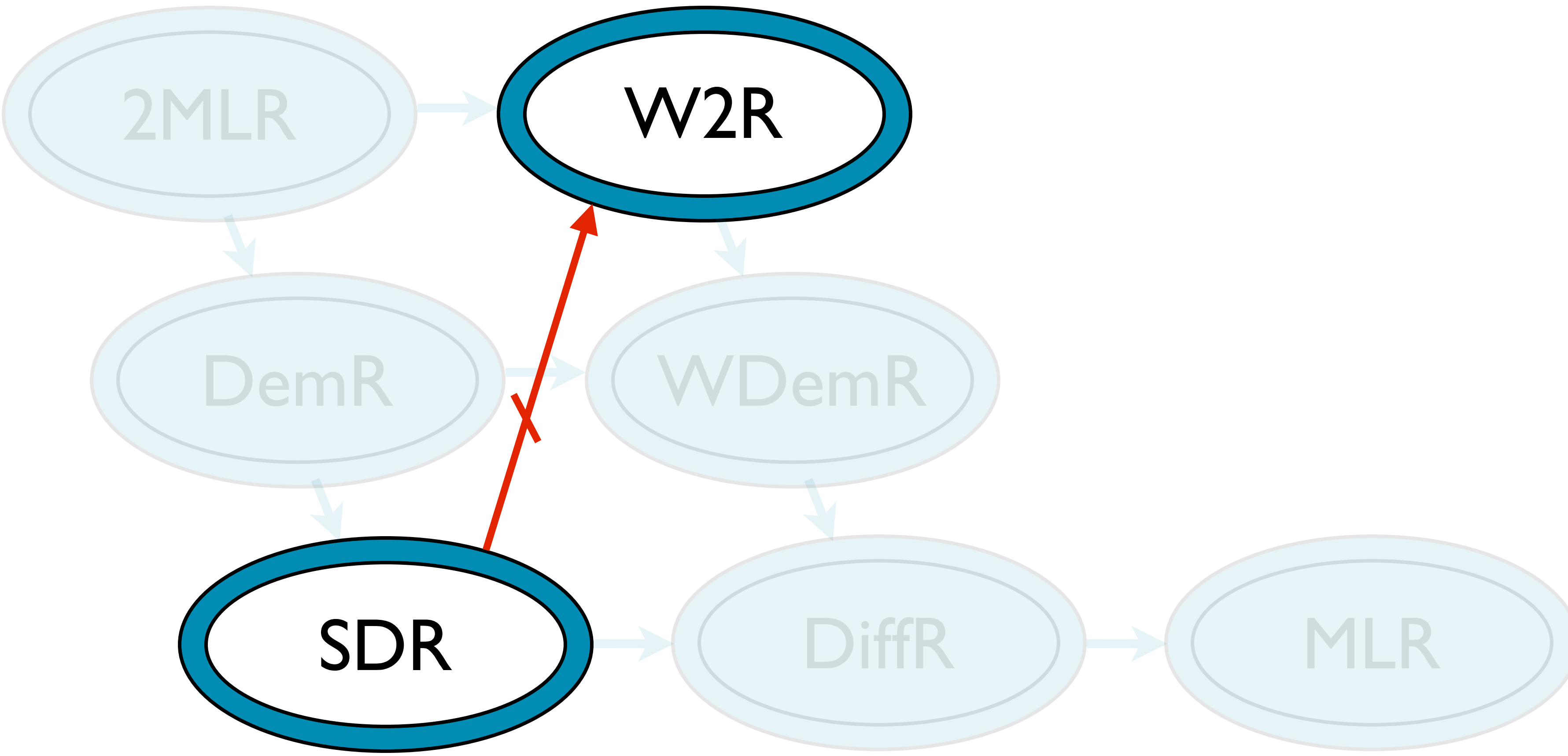


SDR vs DiffR

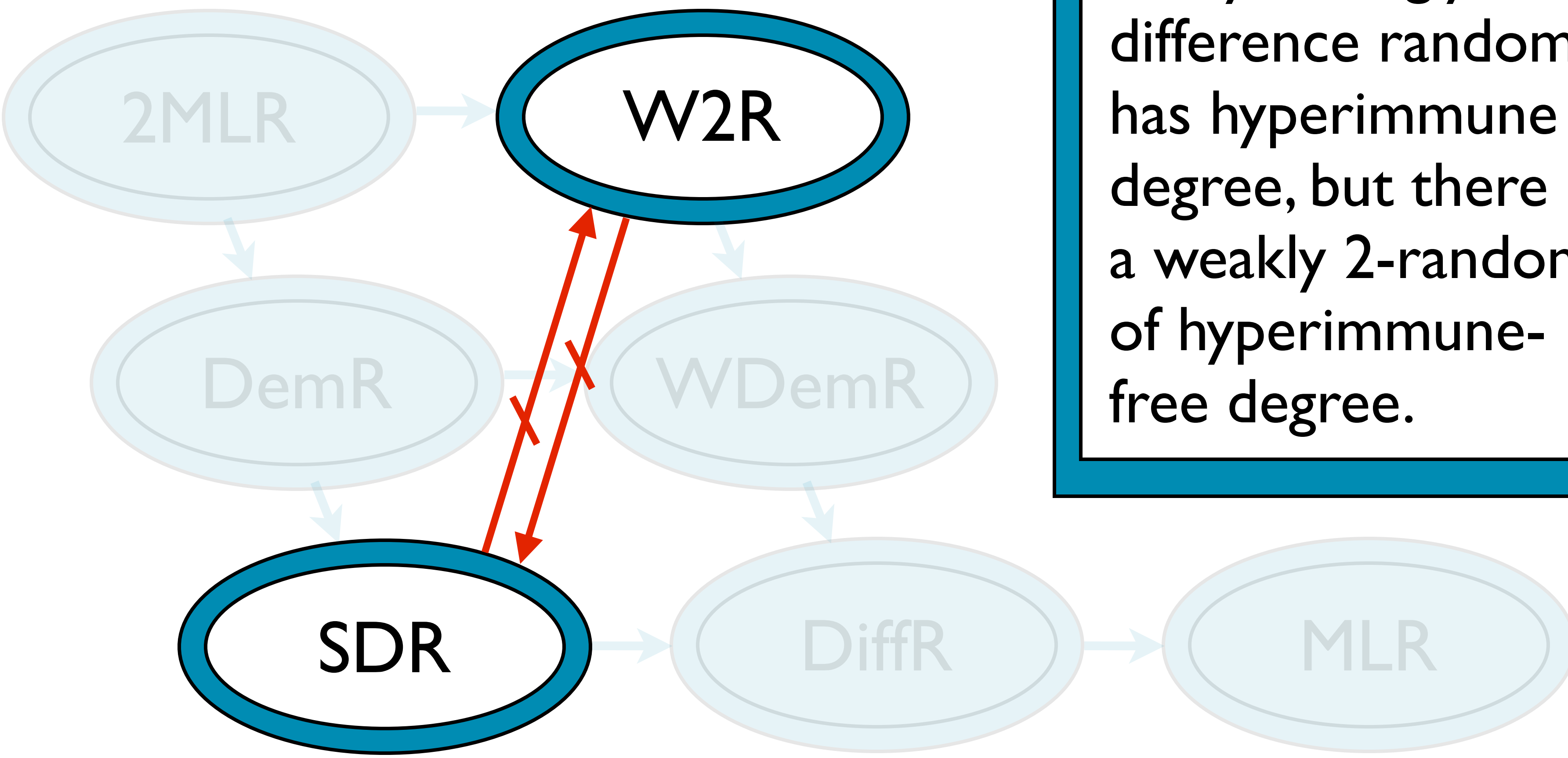


Every strongly difference random has hyperimmune degree, but there is a difference random of hyperimmune-free degree.

SDR vs W2R



SDR vs W2R



Every strongly difference random has hyperimmune degree, but there is a weakly 2-random of hyperimmune-free degree.

2-randoms and 2-generics

Nies, Stephan, and Terwijn proved that every 2-random forms a minimal pair with every 2-generic by arguing as follows:

- Every non-computable degree below a 2-random is low for Ω.

- No non-computable degree below a 2-generic is low for Ω.

SDR and 2-generics

Our result follows from the following two facts:

- If $X \in 2^\omega$ is strongly difference random, then every X-partial computable function is dominated by a total $\emptyset'$-computable function.

- Every non-computable sequence below a 2-generic computes a function not dominated by any $\emptyset'$-computable function.

On hyperimmunity

Martin's proof that almost every Turing degree is hyperimmune can be recast in terms of a probabilistic algorithm.

We feed our algorithm a randomly generated oracle, and with non-zero probability, the algorithm will produce a function not dominated by any computable function.

SDR and hyperimmunity, I

For positive-measure many $X \in 2^\omega$ we want some $g \leq_T X$ satisfying, for every $n \in \omega$,

$\mathcal{R}_e$: If ϕ_e is total, then $g(n_e) = \phi_e(n_e) + 1$ for some $n_e \in \omega$.

For each requirement, using the oracle, the algorithm may guess that $\phi_e(n_e)\downarrow$ for some witness n_e.

SDR and hyperimmunity, II

There are effectively open sets $\mathcal{U}_e$ and $\mathcal{V}_e$ such that those sequences that guess that $\phi_e(n)\downarrow$ for some witness $n \in \omega$ will enter $\mathcal{U}_e$, and if the guess turns out to be correct, they will later enter $\mathcal{V}_e$.

If $X \in 2^\omega$ incorrectly guesses that $\phi_e(n_e)\downarrow$, we will have

$$X \in \mathcal{U}_e \setminus \mathcal{V}_e.$$

Thus, strong difference randoms will make at most finitely many incorrect guesses, and thus the algorithm will produce the desired function.

Thank you!