Deep Π^0_1 Classes

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A motivating question

Let $\mathcal{P} \subseteq 2^\omega$ be a Π^0_1 class, i.e., an effectively closed subset of 2^ω.

$\mathcal{Q}$: How difficult is it to produce a member of $\mathcal{P}$?

To give a reasonable answer to this question, we need to specify:

(i) What methods of producing sequences are under consideration?

(ii) What measure of difficulty are we using?

Of course, once we settle (i) and (ii), the answer to our original question also depends on the class $\mathcal{P}$.

Our approach

methods of producing sequences	$\rightsquigarrow$	Turing functionals equipped with random oracles
measures of difficulty	$\rightsquigarrow$	negligibility and depth

Outline of the talk

1. Some background on randomness
2. Semi-measures, negligibility, and depth
3. Basic results on negligible and deep classes
4. Examples of deep classes

1. Some background on randomness

Martin-Löf randomness

Definition

- A *Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 (i.e. effectively open) subsets of 2^ω such that for each i,

$$\lambda(\mathcal{U}_i) \leq 2^{-i}.$$

- A sequence $X \in 2^\omega$ *passes the Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

- $X \in 2^\omega$ is *Martin-Löf random*, denoted $X \in \mathrm{MLR}$, if X passes every Martin-Löf test.

Computable measures

Definition

A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu(\llbracket\sigma\rrbracket)$ is computable as a real-valued function.

In other words, μ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that

$$|\mu(\llbracket\sigma\rrbracket) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

From now on we will write $\mu(\sigma)$ instead of $\mu(\llbracket\sigma\rrbracket)$.

Randomness with respect to computable measures

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes every μ-Martin-Löf test.

Stephan's dichotomy theorem

Recall that $X \in 2^\omega$ has *PA degree* if X computes a consistent completion of Peano arithmetic.

In 2002, Frank Stephan proved the following:

Theorem (Stephan)

If a Martin-Löf random has PA degree, it is already Turing complete (i.e., $A \geq_T \emptyset'$).

Difference randomness

Another definition of algorithmic randomness that will be useful for us is known as *difference randomness*.

Definition

- A *difference test* is a uniform sequence $((\mathcal{U}_i, \mathcal{V}_i))_{i\in\omega}$ of pairs of Σ^0_1 classes such that for each i,

$$\lambda(\mathcal{U}_i \setminus \mathcal{V}_i) \leq 2^{-i}.$$

- A sequence $X \in 2^\omega$ *passes the difference test* $((\mathcal{U}_i, \mathcal{V}_i))_{i\in\omega}$ if $X \notin \bigcap_i(\mathcal{U}_i \setminus \mathcal{V}_i)$.
- $X \in 2^\omega$ is *difference random* if X passes every difference test.

Difference randomness and Stephan's theorem

The following theorem is quite surprising:

Theorem (Franklin, Ng)

Let A be Martin-Löf random. Then A is difference random if and only if $A \not\geq_T \emptyset'$.

Combining this with Stephan's theorem yields:

Corollary

Let A be Martin-Löf random. Then A is difference random if and only if A does not have PA degree.

2. Semi-measures, negligibility, and depth

Left-c.e. semi-measures

A *semi-measure* $\rho : 2^{<\omega} \to [0, 1]$ satisfies

- $\rho(\varnothing) = 1$ and
- $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$ for every $\sigma \in 2^{<\omega}$.

We will be particularly interested in *left-c.e.* semi-measures.

A semi-measure ρ is left-c.e. if each value $\rho(\sigma)$ is the limit of a non-decreasing computable sequence of rationals, uniformly in σ.

Induced semi-measures

Recall: A *Turing functional* $\Phi : 2^\omega \to 2^\omega$ is given by a c.e. set of pairs of strings (σ, τ) such that if $(\sigma, \tau), (\sigma', \tau') \in \Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$.

For $\sigma \in 2^{<\omega}$, we define $\Phi^{-1}(\sigma) := \{X \in 2^\omega : \exists n\, (X{\upharpoonright}n, \sigma) \in \Phi\}$.

Proposition

(i) *If Φ is a Turing functional, then λ_Φ, defined by*

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}(\sigma))$$

for every $\sigma \in 2^{<\omega}$, is a left-c.e. semi-measure.

(ii) *For every left c.e. semi-measure ρ, there is a Turing functional Φ such that $\rho = \lambda_\Phi$.*

The universal semi-measure

Levin proved the existence of a *universal* left-c.e. semi-measure.

A left-c.e. semi-measure M is universal if for every left-c.e. semi-measure ρ, there is some $c \in \omega$ such that

$$\rho(\sigma) \leq c \cdot M(\sigma)$$

for every $\sigma \in 2^{<\omega}$.

Universal semi-measures are induced by universal Turing functionals.

For example, the functional Φ defined by

$$\Phi(1^e 0X) = \Phi_e(X)$$

is universal (where $(\Phi_e)_{e\in\omega}$ is an effective listing of all Turing functionals.

The measure derived from a semi-measure

If ρ is a semi-measure, we can define

$$\overline{\rho}(\sigma) := \inf_n \sum_{\tau \succeq \sigma \text{ \& } |\tau| = n} \rho(\tau).$$

One can verify that $\overline{\rho}$ is the largest measure such that $\overline{\rho} \leq \rho$ (but it is not a probability measure in general).

Proposition

If ρ is a left-c.e. semi-measure induced by a Turing functional Φ, then

$$\overline{\rho}(\sigma) = \lambda(\{X \in 2^\omega : X \in \Phi^{-1}(\sigma) \text{ \& } \Phi(X) \text{ is total}\}).$$

Negligible classes

Let M be the universal left-c.e. semi-measure.

Then $\overline{M}$ can be seen as a universal measure (universal for all computable measures, as well as the measures derived from left-c.e. semi-measures).

Definition

$\mathcal{S} \subseteq 2^\omega$ is *negligible* if $\overline{M}(\mathcal{S}) = 0$.

We are particularly interested here in negligible Π^0_1 classes.

The intuition behind negligibility

Let $\mathcal{P}$ be a negligible Π^0_1 class.

$\overline{M}(\mathcal{P}) = 0$ means that the probability of producing some member of $\mathcal{P}$ by means of any Turing functional equipped with any sufficiently random oracle is 0.

To see this, note that

$$\overline{M}(\mathcal{P}) = 0 \text{ if and only if } \lambda\Big(\bigcup_{i\in\omega} \Phi_i^{-1}(\mathcal{P})\Big) = 0.$$

In particular, for each Φ_i, $\lambda(\{X \in \mathsf{MLR} : \Phi_i(X) \in \mathcal{P}\}) = 0$.

Deep classes: the idea

We want a property stronger than negligibility for Π^0_1 classes.

Instead of considering how difficult it is to produce a path through a Π^0_1 class $\mathcal{P}$, we consider how difficult it is to produce an *initial segment* of some path through $\mathcal{P}$, level by level.

Deep classes are the "most difficult" of Π^0_1 classes in this respect.

A few more definitions

Let $\mathcal{P} \subseteq 2^\omega$ be a Π^0_1 class.

Let $T^{ext} \subseteq 2^{<\omega}$ be the set of extendible nodes of $\mathcal{P}$,

$$T^{ext} = \{\sigma \in 2^{<\omega} : [\![\sigma]\!] \cap \mathcal{P} \neq \emptyset\}.$$

Thus T^{ext} is the canonical co-c.e. tree such that $\mathcal{P} = [T^{ext}]$ (the set of infinite paths through T^{ext}).

For each $n \in \omega$, T^{ext}_n consists of all strings in T^{ext} of length n.

(I will write T instead of T^{ext} hereafter.)

Deep classes: the definition

Let $\mathcal{P}$ be a Π^0_1 class, and let T be the canonical co-c.e. tree corresponding to $\mathcal{P}$.

$\mathcal{P}$ is a *deep class* if there is some computable, non-decreasing, unbounded function $h : \omega \to \omega$ such that

$$M(T_n) \leq 2^{-h(n)},$$

where $M(T_n) = \sum_{\sigma \in T_n} M(\sigma)$.

That is, the probability of producing some initial segment of a path through $\mathcal{P}$ is effectively bounded above.

3. Basic results on negligible and deep classes

Members of negligible classes

A few observations:

- If a Π^0_1 class contains a computable member, clearly it cannot be negligible.
- Moreover, if a Π^0_1 class contains a Martin-Löf random member, it cannot be negligible, since any Π^0_1 class with a random member must have positive Lebesgue measure.

In fact, the following holds.

Proposition

Let $\mathcal{P}$ be a negligible Π^0_1 class. Then for every computable measure μ, $\mathcal{P}$ contains no $X \in \mathsf{MLR}_\mu$.

Does the converse hold?

Suppose that $\mathcal{P}$ is a Π^0_1 class such that $\mathcal{P} \cap \mathrm{MLR}_\mu = \emptyset$ for every computable measure μ.

Does it follow that $\mathcal{P}$ is negligible? No.

Theorem (Bienvenu, Porter, Taveneaux)

There is a non-negligible Π^0_1 class $\mathcal{P}$ such that $\mathcal{P} \cap \mathrm{MLR}_\mu = \emptyset$ for every computable measure μ.

Idea: Downey, Greenberg, and Miller construct a non-negligible, perfect thin Π^0_1 class. Extending a result of Simpson's, we show that every perfect thin class has μ-measure 0 for every computable measure μ.

Depth vs. negligibility

It's clear that every deep class is negligible. Is every negligible class deep? Again, no.

Theorem (Bienvenu, Porter, Taveneaux)

There is a negligible class $\mathcal{P}$ that is not deep.

Idea: Define a co-c.e. tree T and a left-c.e. semi-measure ρ such that

(i) $M(T_{f(n)}) \leq 2^{-n}$ for some fast-growing $f : \omega \to \omega$ (to ensure negligibility), and

(ii) $\rho(T_{f(n)}) \geq 2^{-h(n)}$ for some computable order h (to ensure non-depth).

We use a finite injury argument to carry this out.

Why use the co-c.e. tree in the definition of depth?

For every Π^0_1 class $\mathcal{P}$ there is a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{P} = [T]$.

Why can't we use this computable tree T in the definition of depth?

In general, T will contain non-extendible nodes, so even if we can compute some element in T_n, we still may fail to compute an initial segment of a member of $\mathcal{P}$.

Can we give a better reason to restrict our attention to the canonical co-c.e. tree?

Vindicating the definition, 1

Theorem (Bienvenu, Porter, Taveneaux)

Let T be a computable tree. Then there is no computable order h such that $M(T_n) \leq 2^{-h(n)}$ for every $n \in \omega$.

Proof.

Suppose $M(T_n) \leq 2^{-h(n)}$ for some computable order h.

Case 1: T has only finitely many non-extendible nodes.

Then the leftmost path X of T is computable (since T is computable).

It follows that δ_X, the Dirac measure on X, is computable.

Thus there is some c such that $M(T_n) \geq 2^{-c}\delta_X(T_n) = 2^{-c}$, contradicting our assumption.

Vindicating the definition, 2

Case 2: T has infinitely many non-extendible nodes.

We define a computable sequence of terminal nodes $(\sigma_i)_{i\in\omega}$ and a computable function $f : \omega \to \omega$ such that

- f is strictly increasing, and
- $|\sigma_i| = f(i)$ for every i.

We define a semi-measure ρ such that $\rho(\sigma_n) = 2^{-K(n)}$ for every n (consistently extending ρ to initial segments of each σ_n), where $K(n)$ is the prefix-free Kolmogorov complexity of n.

Then there is some c such that
$M(T_{f(n)}) \geq 2^{-c}\rho(T_{f(n)}) \geq 2^{-K(n)-c}$.

But then by our assumption, $2^{-K(n)-c} \leq 2^{-h(f(n))}$, and hence $h(f(n)) \leq K(n) + c$.

This contradicts the fact that there is no computable lower bound for K.

Randoms computing members of deep classes

Which random sequences can compute some member of a deep class?

Note that if X has PA degree and hence can compute a member of every deep class.

Thus, by Stephan's dichotomy theorem, if $X \in \mathrm{MLR}$ and $X \geq_T \emptyset'$, X computes some member of a deep class.

But this is the best we can do.

Theorem (Bienvenu, Porter, Taveneaux)

No difference random sequence can compute a member of a deep class.

4. Examples of deep classes

Consistent completions of Peano arithmetic

The following is implicit in work of Levin and Stephan.

Theorem

The Π^0_1 class of consistent completions of PA is a deep class.

Equivalently, we can consider the class $\mathcal{P}$ of total extensions of a universal partial computable $\{0,1\}$-valued function.

Let $u(\langle e, x\rangle) = \phi_e(x)$, where $(\phi_e)_{e\in\omega}$ is an effective enumeration of all partial computable $\{0,1\}$-valued functions.

We will define a partial computable $\{0,1\}$-valued function ϕ_e (where we know e in advance by the recursion theorem), and this will allow us to show that $\mathcal{P}$ is deep.

The proof idea, 1

Since we are defining ϕ_e, we have control of the values $u(\langle e, x\rangle)$ for every $x \in \omega$.

Let $(I_k)_{k\in\omega}$ be an effective collection of intervals forming a partition of ω, where we have control of 2^{k+1} values of u inside of I_k for each $k \in \omega$.

Step 1: For each k, we consider the sets

$$E_{k,s} = \{\sigma \in 2^{<\omega} : \sigma{\upharpoonright}I_k \text{ extends } u_s{\upharpoonright}I_k\},$$

and wait for a stage s such that

$$M(E_{k,s}) \geq 2^{-k}.$$

The proof idea, 2

Step 2: Pick some $y \in I_k$ on which we have yet to define u.

Consider the sets

$$E^0_{k,s}(y) = \{\sigma \in E_{k,s} : \sigma(y) = 0\}$$

and

$$E^1_{k,s}(y) = \{\sigma \in E_{k,s} : \sigma(y) = 1\}.$$

Then $M(E^i_{k,s}(y)) \geq 2^{-(k+1)}$ for $i = 0$ or 1 (or both).

If this holds for $i = 0$, we set $u(y) = 1$; otherwise we set $u(y) = 0$.

The proof idea, 3

We repeat the process, going back to Step 1.

We can repeat the process at most 2^{k+1} times (since we have enough values to work with in I_k).

Eventually, we will get stuck at Step 1.

Setting $f(k) = \max(I_k)$, we will have

$$M(\{\sigma : \sigma{\upharpoonright}f(k) \text{ extends } u\}) \leq 2^{-k}.$$

That is,

$$M(T_{f(k)}) \leq 2^{-k}.$$

Shift-complex sequences

For $\delta \in (0, 1)$ and $c \in \omega$, we say that $X \in 2^\omega$ is *(δ, c)-shift complex* if

$$K(\tau) \geq \delta|\tau| - c$$

for every subword τ of X.

The following draws upon work of Rumyantsev.

Theorem (Bienvenu, Porter, Taveneaux)

For every $\delta \in (0, 1)$ and $c \in \omega$, the (δ, c)-shift complex sequences form a deep class.

DNR$_h$ functions

Let h be a computable order.

f is a DNR$_h$ function if

- f is total,
- $f(e) \neq \phi_e(e)$ for every e, and
- $f(e) < h(e)$ for every e.

Theorem (Bienvenu, Porter, Taveneaux)
DNR$_h$ is a deep class if and only if $\sum_{n=0}^{\infty} \frac{1}{h(n)} = \infty$.