Randomness and Accessible Objects in Mathematics

Christopher P. Porter
Université Paris 7
LIAFA

FPMW5
Clermont-Ferrand
17 October 2013

Introduction

The purpose of this talk is to delineate and respond to a specific problem that has arisen in the context of a debate about the adequacy of certain formalizations of the concept of randomness.

Certain definitions of randomness, which I call *extensional definitions* of randomness, face a problem that I call *the problem of unruly instances*:

For each such definition $\mathscr{D}$, there is some collection $\mathcal{U}$ of objects such that

(i) each object $x \in \mathcal{U}$ is $\mathscr{D}$-random, but

(ii) there are grounds for holding that each $x \in \mathcal{U}$ should not be counted as random (that is, random according to certain widely held, pre-theoretic intuitions about the concept of randomness).

Introduction, continued

That is, even though each $x \in \mathcal{U}$ is *formally* random, it is disputable whether x should be counted as *intuitively* random, at least according to certain ways of understanding what it means for an object to be intuitively random.

I refer to these objects to as *unruly instances* of formal randomness.

Introduction, continued

As I hope to clarify, the problem of unruly instances is a problem on two different approaches to understanding the various definitions of randomness:

- *The singular approach*, which seeks to identify one definition of randomness that captures the intuitive conception of randomness; and
- *The plural approach*, according to which multiple extensionally distinct definitions of randomness are necessary to capture everything that mathematicians have held to be significant about the notion of randomness.

However, as I will argue, only the plural approach can adequately address the problem.

My goals for today

The goals of this talk are the following:

- to explain what is particularly problematic about these unruly instances for both the singular and plural approaches;
- to briefly outline Borel's account of accessible and inaccessible objects in mathematics; and
- to bring Borel's account to bear on the problem of unruly instances.

Outline of the talk

1. Delineating the problem of unruly instances
2. Examples of unruly instances
3. Borel's account of accessible and inaccessible mathematical objects
4. Applying Borel's account of accessibility

1. Delineating the problem of unruly instances

Extensional definitions of randomness

Before I explain the problem of unruly instances, I need to specifiy the definitions for which this problem is a problem.

The formalizations of randomness that face this problem are what I call *extensional*.

Roughly, a definition of randomness is extensional if it counts an object as random if it satisfies certain properties that are satisfied by the typical outcomes of some paradigmatically random process (such as the repeated tosses of an unbiased coin).

By contrast, a definition of randomness is *intensional* if it counts an object as random if it is produced by some paradigmatically random process such as the repeated tosses of a fair coin.

Some restrictions

The extensional definitions of randomness that I will consider here have the following properties:

1. The objects to which these definitions are applicable are infinite binary sequences (i.e. members of 2^ω).

2. Each definition $\mathscr{D}$ can formulated in terms of a countable collection of properties $\{\Phi_i\}_{i\in\omega}$, such that
   - for each $i \in \omega$, the set $\{X \in 2^\omega : \Phi_i(X)\}$ has Lebesgue measure one, and
   - $X \in 2^\omega$ is $\mathscr{D}$-random if and only if $\Phi_i(X)$ for every $i \in \omega$.

3. Each definition $\mathscr{D}$ yields a partition of 2^ω into two classes, the $\mathscr{D}$-random sequences and the non-$\mathscr{D}$-random sequences.

What's the problem?

Why do unruly instances pose a problem for extensional definitions of randomness?

The general worry is that these unruly instances might undermine the claim of a definition of randomness to being an adequate definition of randomness.

Definitional inadequacy

More precisely, the worry is that one might be justified in reasoning as follows:

(1) Definition $\mathscr{D}$ counts as random some object with property $\mathcal{P}$.

(2) Any object with property $\mathcal{P}$ should not be counted as random.

(3) Any definition of randomness that counts as a random a sequence that should not be counted as random is not an adequate definition of randomness.

(C) Therefore $\mathscr{D}$ is not an adequate definition of randomness.

Henceforth, let us refer to arguments of the above form as *definitional inadequacy arguments*.

What counts as an unruly instance?

Definitional inadequacy arguments appear, at least implicitly, scattered throughout the philosophical literature on algorithmic randomness.

But what is often lacking in discussions of these arguments is a general account as to what qualifies as an unruly instance.

Despite the lack of such an account, unruly instances still threaten to undermine the claim of adequacy of the various definitions of randomness on both the singular and the plural approaches to the definitions of randomness.

The challenge for each approach

Let $\mathscr{D}$ be a fixed definition of randomness, and suppose that there is an unruly instance x of $\mathscr{D}$-randomness.

Singular approach: Those who hold that $\mathscr{D}$ captures the intuitive conception of randomness must answer the question:

- How can $\mathscr{D}$ capture the intuitive conception of randomness while counting x as random?

Plural approach: Those who hold that $\mathscr{D}$ is one of many definitions that capture some mathematically significant random phenomena must answer the question:

- How can $\mathscr{D}$ capture the mathematically significant random phenomena while counting x as random?

2. Examples of unruly instances

Example 1: Normal sequences

For $\sigma, \tau \in 2^{<\omega}$, let $\#_\sigma(\tau)$ denote the number of occurrences of the string σ in the string τ.

For example, $\#_{101}(01010110) = 2$.

A sequence $X \in 2^\omega$ is *normal* in base 2 if for every $\sigma \in 2^{<\omega}$,

$$\lim_{n\to\infty} \frac{\#_\sigma(X\restriction n)}{n} = 2^{-|\sigma|}.$$

Further, $X \in 2^\omega$ is *absolutely normal* if X is normal in base b for every $b \in \omega$.

Example 1: Normal sequences, continued

For many mathematical purposes, normal sequences are considered to be sufficiently random.

For instance, in number-theoretic contexts, one finds the question "Is π random?", which is taken to mean "Is π is absolutely normal?" or even "Is π normal in some fixed base b?"

Normality can be viewed as the weakest notion of algorithmic randomness:

- $X \in 2^\omega$ is normal in base 2 if and only if X is incompressible by a lossless finite-state compressor (Becher, Heiber).
- $X \in 2^\omega$ is normal in base 2 if and only if X is stochastic with respect to selection by a finite state automaton (Agafonov).

Example 1: Normal sequences, continued

However, there are unruly instances of normal sequences, such as Champernowne's number, which is normal in base 2.

$$1101110010111011110001001101010111100110111101111\ldots$$

One can even construct a computable sequence that is absolutely normal.

Example 1: Normal sequences, continued

However, there are unruly instances of normal sequences, such as Champernowne's number, which is normal in base 2.

1 10 11 100 101 110 111 1000 1001 1010 1011 1100 1101 1110 1111 . . .

One can even construct a computable sequence that is absolutely normal.

Example 2: Church stochastic sequences

An important definition in the development of algorithmic randomness is what is nowadays referred to as *Church stochasticity*.

The definition is due to von Mises and Church (1940).

$X \in 2^\omega$ is Church stochastic if

(i) it satisfies the law of large numbers (i.e. has relative limiting frequency of 0 and 1 equal to $1/2$), and

(ii) every subsequence selected from X by a computable place selection also satisfies the law of large numbers.

Example 2: Church stochastic sequences, continued

Church stochastic sequences are much more well-behaved than normal sequences.

For instance, no computable sequence is Church stochastic.

Unruly instance: Ville proved that there is a Church stochastic sequence that fails to satisfy the law of the iterated logarithm.

Such a sequence has more 0s than 1s in every initial segment, even though in the limit, the relative frequencies of 0 and 1 both converge to $1/2$.

Example 3: Martin-Löf random sequences

Martin-Löf randomness is a notion of randomness given in terms of computably enumerable statistical tests.

- To test the hypothesis that a given infinite sequence has random origin (say, produced by the tosses of a fair coin), we test the hypothesis at all levels of significance of the form 2^{-n}.
- A sequence passes the test if it is not contained in at least one of the critical regions.
- A sequence is Martin-Löf random if it passes all such tests.

Martin-Löf: this definition "satisfies all intuitive requirements"

Example 3: Martin-Löf random sequences, continued

Martin-Löf randomness captures multiple instances of "almost-everywhere" phenomena from classical mathematics.

That is, for a number of theorems that assert that some condition is true for almost every point in a given domain (i.e. measure one many points), one can show that the result, suitably "effectivized", holds at *all* and *only* the Martin-Löf random points.

- Analysis: differentiability of computable real-valued functions of bounded variation (Brattka, Miller, Nies).
- Ergodic theory: Birkhoff's Ergodic Theorem with respect to effectively closed classes.
- Information theory: an effective version of the Shannon-McMillan-Breiman theorem (Hoyrup).

Example 3: Martin-Löf random sequences, continued

However, there are a number of unruly instances of Martin-Löf random sequences:

- Chaitin's Ω is a Martin-Löf random sequence that encodes the halting problem and has a number of other strong properties that seem to be incompatible with being random.
- Δ^0_2 sequences: There are Martin-Löf random sequences that are decidable in the limit by a trial and error procedure.
- Kučera-Gács: Every sequence is Turing reducible to a Martin-Löf random sequence.

3. Borel's account of accessible and inaccessible mathematical objects

Borel's *Les Nombres Inaccessibles*

In his 1952 monograph *Les Nombres Inaccessibles*, Émile Borel provides an account of the distinction between accessible and inaccessible mathematical objects.

His account is an interesting one that merits more attention than it has received, but it also has some relevance for the topic at hand.

Borel on the science of the accessible

It seems to me that mathematicians as well, while maintaining the full right to work out abstract theories deduced from arbitrary non-contradictory axioms, have an interest in distinguishing, among the objects of thought which are the substance of their science, those which are truly accessible, that is to say, have an individuality, a personality, which characterizes them without ambiguity.

One is thus led to define in a precise manner a science of the accessible and of the real, beyond which it remains possible to develop a science of the imaginary and of the imagined, these two sciences being able, in certain cases, to lend each other mutual support. (Les Nombres Inaccessibles, pp. ix-x, translation by Bagemihl)

Accessible vs. inaccessible numbers

Borel attempts to develop his "science of the accessible" by first discussing accessible and non-accessible numbers:

> *When we say that a process has allowed us to define a determinate integer, we mean that we set clear and precise rules so that any mathematician knows which integer we have defined and that two different mathematicians, when speaking about this integer, know that there is no misunderstanding between them, that is to say they are certain that the number designated by the letter n is the same for one as it is for the other. (Ibid., p. 1)*

Some features of Borel's account

Borel's considers two notions of accessibility, relative accessibility and absolute accessibility:

> *We have defined the relatively accessible numbers as those which may have been or may be effectively defined by any human, before humanity disappears [...] (Ibid., p. 13)*

Absolutely accessible numbers include all natural numbers, rational numbers, algebraic numbers, and numbers that can be derived from other accessible numbers (even by means of infinitary operations).

Some features of Borel's account, continued

Moreover, the boundary between accessible and inaccessible objects is also inaccessible. For instance, concerning natural numbers, Borel writes,

> *Our conclusion is that there are inaccessible whole numbers, that is to say that they will never be achieved by any human, but that by their very definition, we do not know them and it is impossible for us to indicate the point at which the integers are inaccessible, since this limit is itself inaccessible. (Ibid., p. 4)*

Accessible vs. inaccessible sets

For our purposes, Borel's account of accessible and inaccessible sets is particularly noteworthy.

First, the definition of an accessible set is very similar to that of an accessible natural number.

> *We say thus that a set is accessible when it can be defined in such a manner that two mathematicians, when they are speaking of it, are certain that they are speaking of the same set. (Ibid., p. 104)*

Accessible vs. inaccessible sets, continued

However, there is an interesting additional wrinkle in Borel's discussion of accessible sets:

> *For a set to be accessible, we should not require that all of its points are accessible, otherwise the continuum itself would be considered as inaccessible and we could even regard any infinite set as inaccessible, even if it is countably infinite, since such a set contains points that are, in fact, inaccessible. (Ibid. p. 104)*

Thus, a set can be counted as accessible even if contains inaccessible members.

Another example of such a set is the Cantor middle-third set.

A few remarks

More precisely, a mathematical object x is accessible if there is some linguistic expression ϕ such that any competent user of the expression ϕ knows that ϕ refers to x and not some other object.

There are many questions about Borel's account that we'll have to set aside for now. (For instance, how does competence in using ϕ lead to knowledge of the reference of ϕ?)

Note that Borel does not require that ϕ be some formal linguistic expression.

Rather, given that Borel's concern is with something along the lines of *human* accessibility, it's not unreasonable to consider ϕ as being a natural language expression.

A few remarks, continued

Borel's notion of accessibility can thus be seen as a weak notion of definability, something along the lines of natural language definability.

In his review of Borel's book, the mathematician Bagemihl writes,

> *Borel's notion of accessibility, although of heuristic significance, seems too subjective, temporal, and, by precluding intrinsically the possibility of delimiting the realm of the accessible, vague, according to his own standards, to "define in a precise manner a science of the accessible and of the real." (Review of Les Nombres Inaccessibles, p. 409)*

These weaknesses notwithstanding, Borel's account can still do some work for us.

4. Applying Borel's account of accessibility

Unruly instances and Borel's account

Now I'd like to consider the extent to which Borel's account of accessible and inaccessible objects can shed light on the problem of unruly instances.

Specifically, it can help us answer the question: What counts as an unruly instance?

Unruly instances and accessible null sets

In the first place, what the unruly instances that are identified in the algorithmic randomness literature have in common is that each belongs to a set of measure zero that is accessible in Borel's sense.

That is, an unruly instance corresponds to a property that is both rare (as only measure zero many sequences satisfy it) and unambiguously describable.

Defining randomness in terms of null sets

As I discussed earlier, each extensional definition of randomness $\mathscr{D}$ can be formulated in terms of a countable collection of properties $\{\Phi_i\}_{i\in\omega}$, such that

- for each $i \in \omega$, the set $\{X \in 2^\omega : \Phi_i(X)\}$ has Lebesgue measure one, and
- $X \in 2^\omega$ is $\mathscr{D}$-random if and only if $\Phi_i(X)$ for every $i \in \omega$.

Equivalently, we can consider the properties $\{\neg\Phi_i\}_{i\in\omega}$, so that each $\neg\Phi_i$ defines a set of measure 0 (which I will hereafter refer to as *null properties*).

If X satisfies $\neg\Phi_i$, then it is disqualified from being counted as $\mathscr{D}$-random.

A modest sharpening

For a given definition $\mathscr{D}$, formulated in terms of a countable collection of null properties $\{\neg\Phi_i\}_{i\in\omega}$, an unruly instance of $\mathscr{D}$-randomness corresponds to an accessible null property $\neg\Psi$ that is not equivalent to any null property $\neg\Phi_i$.

Of course, this gloss on unruly instances is unclear, insofar as it draws upon Borel's unclear notion of accessibility.

A difficulty for the singular approach

In the context of evaluating whether a definition of randomness is intuitively adequate, this unclarity isn't really a problem.

If our target in formalizing randomness is the collection of intuitively random sequences, understood as the collection of sequences that an ideal human would judge to be random, then intuitively random sequences should avoid all humanly accessible null sets.

But if Borel is correct that the boundary between accessible and inaccessible objects is itself inaccessible, then there seem to be no conditions under which we could recognize that a given definition captures the intuitively random sequences.

What about the plural approach?

This difficulty is avoided on the plural approach to definitions of randomness.

For on this approach, the adequacy of a given definition of randomness is not determined by how well it matches up with some pre-theoretic conception of randomness.

Instead, on the plural approach, adequacy is relative to certain purposes.

Purposes

Different definitions of randomness serve different purposes: often for a given purpose, there is a degree of randomness necessary to carry out that purpose (in some cases, we can even show such a degree is necessary and sufficient).

Disqualification is not an absolute notion; what might be considered unruly instances with respect to one purpose might not be considered as unruly instances with respect to another.

An example

Earlier we saw that Martin-Löf random sequences correspond precisely to the points that satisfy Birkhoff's ergodic theorem for effectively closed classes.

The fact that some Martin-Löf random sequences are Δ^0_2 has no bearing on whether or not the above result holds.

However...

I think there's much more to say on this matter.

In fact, Borel's account of accessibility and inaccessibility, suitably sharpened, may be of help to address the problem of unruly instances on the plural approach to definitions of randomness.

This is still work in progress, but I think there is some promise to this approach.

An agent-centric approach to randomness

It is helpful to think of a definition of randomness as being given in terms of an agent who has a specific set of resources for testing whether or not a given sequence is random.

For instance, Martin-Löf randomness corresponds to an agent who only has access to computably enumerable statistical tests.

On this account, the set of Δ^0_2 sequences is inaccessible to this agent: her resources do not allow her to produce a test that is passed only by non-Δ^0_2 sequences.

An agent-centric approach to randomness, continued

Similarly, the collection of normal sequences corresponds to an agent with only very weak computational resources at his disposal (roughly, those given by finite state automata).

Moreover, the singleton set consisting of Champernowne's number is inaccessible to this agent: his resources do not allow him to recognize the pattern in Champernowne's number, a pattern that is obvious to us.

There's still much to develop here, but such an account may help us better understand why it is that a definition of randomness can successfully fulfill some purpose despite the presence of unruly instances.