Effective Notions of Typicality

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Introduction

The aim of this talk is to outline a logical approach to arbitrary/generic objects in terms of two different effective notions of typicality:

- effective randomness, and
- effective genericity.

Outline

1. Fine's Account of Arbitrary Objects
2. Measure and Category
3. Effective Randomness and Effective Genericity
4. Effectively Typical Objects as Generic Objects

1. Fine's Account of Arbitrary Objects

Motivating Fine's account

In *Reasoning with Arbitrary Objects*, Kit Fine develops an account of arbitrary objects with the aim of providing a "satisfactory explanation" of certain inferences, most notably, universal generalization.

Fine is particularly interested in providing a formal account of arbitrary objects in which he can formalize inferences such as those of the following form.

> *Let* n *be an arbitrary natural number.*
>
> $$\vdots$$
>
> *Thus* n *has property* P*. But* n *was arbitrary, and thus every natural number has* P*.*

The principle of generic attribution

In Fine's framework, arbitrary objects function as a kind of variable, as each arbitrary object has a range of values.

Central to his account is what he calls the *principle of generic attribution* (PGA): "each arbitrary object should have the properties common to the individuals in its range."

Naive formulation: For a property P, P holds of an arbitrary object a if and only if P holds of every individual object in the range of a.

An initial problem

One problem with this naive formulation: If we choose the property P to be "is an individual object," then by the naive version of the PGA, it follows that every arbitrary object is an individual object.

Fine's solution: restrict PGA to hold only for *generic properties*.

- Properties such as "being an odd number" are generic.
- Properties such as "being an individual object" are not generic.

The generic property problem

However, Fine does not precisely specify which properties are the generic ones.

According to Fine, while we may not know in advance which properties will lead to a contradiction (such as “being an individual object”), many languages “of natural and independent interest” only make use of generic predicates, so PGA will be applicable to these languages.

Tennant's response:

> *'Being generic' ought to be a decidable property of conditions expressible in the language. Only then will the principle of generic attribution have application of sure axiomatic status.*

Let us refer to this problem of identifying the generic properties as the *generic property problem*.

An alternative approach

Regardless of the merits of Fine's account, it cannot accommodate all of the uses of generic or arbitrary objects in mathematics (as Fine would readily admit).

It is not the case that every notion of generic object in mathematics is intended to license the attributions of properties to *every* object in a given domain.

With the examples that we will consider today, the generic/arbitrary objects in question are

1. generic with respect to a specific collection of properties;
2. these properties do not hold of every element of the relevant domain; but
3. they do hold of *most* elements in the domain.

I will motivate these examples as putative responses to another version of the generic property problem.

2. Measure and Category

The basic picture

There are certain 'typical' properties that, although not satisfied by every object in a given collection, are satisfied by most objects in that collection.

These 'typical' properties are defined in terms of various notions of largeness.

There are two notions of largeness that I will consider today, given in terms of:

- measure;
- category.

For simplicity, let us restrict our attention to the space 2^ω (and sometimes $[0, 1]$).

Measure in 2^ω

For $\sigma \in 2^{<\omega}$, we let $[\![\sigma]\!] = \{X \in 2^\omega : \sigma \prec X\}$.

Then the Lebesgue measure on 2^ω is defined by first setting

$$\lambda([\![\sigma]\!]) = 2^{-|\sigma|}$$

for all $\sigma \in 2^{<\omega}$ and then extending λ to all measurable subsets of 2^ω in the standard way.

Measure in 2^ω (continued)

Let $\mathcal{X} \subseteq 2^\omega$.

$\mathcal{X}$ is small with respect to measure $\approx$ $\lambda(\mathcal{X}) = 0$ ("$\mathcal{X}$ is a nullset")

$\mathcal{X}$ is large with respect to measure $\approx$ $\lambda(\mathcal{X}) = 1$

Category in 2^ω

Let $\mathcal{X} \subseteq 2^\omega$.

- $\mathcal{X}$ is *nowhere dense* if the closure of $\mathcal{X}$ has empty interior.
- $\mathcal{X}$ is *meager* if it can be written as the countable union of nowhere dense sets.
- $\mathcal{X}$ is *comeager* if $\mathcal{X}^c$ is meager.

$\mathcal{X}$ is small with respect to category $\approx$ $\mathcal{X}$ is meager

$\mathcal{X}$ is large with respect to category $\approx$ $\mathcal{X}$ is comeager

The significance of measure and category

By means of the notions of measure and category, we can identify certain properties that, although not universal, are "almost universal."

More specifically, for some property $\mathcal{P}$, if we know that

$$\{X \in 2^\omega : \mathcal{P}(X)\}$$

is large with respect to measure or with respect to category, then although we are not justified in concluding that this property holds for every member of 2^ω, we are justified in holding that the failure to satisfy $\mathcal{P}$ is atypical behavior for members of 2^ω.

The significance of measure and category (continued)

This, in turn, gives us a degree of control over the exceptions to $\mathcal{P}$.

For instance, in analysis it is common to identify two functions whose values differ only on a set of measure zero.

Note, however, that even if one knows that the satisfaction of some property $\mathcal{P}$ is typical for members of 2^ω, in general this gives us no information about which members of 2^ω fail to satisfy $\mathcal{P}$.

Thus, by means of measure and category, we can identify various kinds of typical behavior, but it is an *incomplete* specification of typicality.

That is, we identify $\mathcal{P}$ as a typical property without specifying which points are the typical points.

Two examples

Theorem

A function $f : [0,1] \to \mathbb{R}$ *of bounded variation is differentiable on a set of measure one.*

Theorem (Bruckner, Leonard 1966)

A set $\mathcal{X} \subseteq [0,1]$ *is the set of discontinuities of the derivative of some differentiable function* $f : [0,1] \to \mathbb{R}$ *if and only if* $\mathcal{X}$ *is a meager* F_σ *set.*

A definition of generic objects?

We have yet not arrived at the desired alternative account of generic objects.

What about the following approach?

Let us say that

- $Y \in 2^\omega$ is generic with respect to measure if it is contained in every $\mathcal{X} \subseteq 2^\omega$ of measure one, and
- $Y \in 2^\omega$ is generic with respect to category if it is contained in every comeager $\mathcal{X} \subseteq 2^\omega$.

Not yet...

Problem: these two notions of generic object are empty.

For each $X \in 2^\omega$, the set

$$\{Y \in 2^\omega : Y \neq X\}$$

has measure one and is comeager.

Thus, we cannot require that the objects that are generic with respect to measure or category satisfy *all* large properties.

We need to restrict the large properties in some way; that is, we need to identify something along the lines of generic properties.

That is, we face another version of the generic property problem.

3. Effective Randomness and Effective Genericity

Countable collections of properties

Note that if we define generic objects in terms of a countable collection of properties, the resulting definition will have a non-empty extension:

- The countable intersection of a collection of sets of Lebesgue measure one has Lebesgue measure one.
- The countable intersection of a collection of comeager sets is comeager.

In what follows, I will identify various countable collections of properties that yield several families of notions of effective typicality.

Arithmetical subsets of ω

Let $S \subseteq \omega$.

- S is a Σ^0_1 set if there is some computable predicate $P(x,y)$ such that
$$S = \{x \in \omega : (\exists y)\ P(x,y)\}$$
- S is a Π^0_1 set if S^c is a Σ^0_1 set.

Arithmetical subsets of ω, continued

- S is a Σ^0_n set if there is some computable predicate $P(x, y_1, \dots y_n)$ such that

  $$S = \{x \in \omega : (\exists y_1)(\forall y_2)\dots(Qy_n)\ P(x, y_1, \dots y_n)\}$$

  where Q is "$\forall$" if n is even and Q is "$\exists$" if n is odd.
- S is a Π^0_n set if S^c is a Σ^0_n set.

We can define Σ^0_n and Π^0_n subsets of $2^{<\omega}$ simply by identifying ω and $2^{<\omega}$.

Effective genericity 1: weak n-genericity

A set $S \subseteq 2^{<\omega}$ is dense if for every $\sigma \in 2^{<\omega}$, there is some $\tau \in S$ such that $\tau \succeq \sigma$.

Definition

Let $n \geq 1$. $X \in 2^{\omega}$ is weakly n-generic if for every dense Σ^0_n $S \subseteq 2^{<\omega}$, there is some $\tau \in S$ such that $\tau \prec X$.

Effective genericity 2: n-genericity

Definition

Let $n \geq 1$. $X \in 2^\omega$ is n-generic if for every Σ^0_n $S \subseteq 2^{<\omega}$, there is some $\tau \prec X$ such that

(i) either $\tau \in S$, or

(ii) for all $\sigma \succeq \tau$, $\sigma \notin S$.

Some remarks

1. Note that we can replace the Σ^0_n sets with $\Sigma^{0,\emptyset^{(n-1)}}_1$ sets, where
$$\emptyset^{(k+1)} = \{e : \phi_e^{\emptyset^{(k)}}(e)\downarrow\}.$$
2. For $n \geq 1$, the following holds:
   weak $(n+1)$-genericity $\Rightarrow$ n-genericity $\Rightarrow$ weak n-genericity
3. The reverse implications do not hold.

Arithmetical subsets of 2^ω

Let $\mathcal{X} \subseteq 2^\omega$.

- $\mathcal{X}$ is Σ^0_1 class if there is some computable predicate $P(x) \subseteq 2^{<\omega}$ such that

$$\mathcal{X} = \{Y \in 2^\omega : (\exists n)\ P(Y{\upharpoonright}n)\}$$

- $\mathcal{X}$ is Π^0_1 class if S^c is a Σ^0_1 class.

Arithmetical subsets of 2^ω

- $\mathcal{X}$ is a Σ^0_n class if there is some computable predicate $P(y_1, \ldots, y_n)$ such that

  $$\mathcal{X} = \{Y \in 2^\omega : (\exists y_1)(\forall y_2)\ldots(Qy_n)\ P(y_1, \ldots, y_{n-1}, Y \restriction y_n)\}$$

  where Q is "$\forall$" if n is even and Q is "$\exists$" if n is odd.

- $\mathcal{X}$ is a Π^0_n class if $\mathcal{X}^c$ is a Σ^0_n class.

Effective randomness 1: weak n-randomness

Definition

Let $n \geq 1$. $Y \in 2^\omega$ is weakly n-random if for every Σ^0_n $\mathcal{X} \subseteq 2^\omega$ such that $\lambda(\mathcal{X}) = 1$, we have $Y \in \mathcal{X}$.

Effective randomness 2: n-randomness

Definition

Let $n \geq 1$. $Y \in 2^\omega$ is n-random if for every sequence of uniformly $\Sigma_1^{0,\emptyset^{(n-1)}}$ classes $(\mathcal{U}_i)_{i\in\omega}$ such that

$$\lambda(\mathcal{U}_i) \leq 2^{-i},$$

we have $Y \notin \bigcap_{i\in\omega} \mathcal{U}_i$.

Some remarks

1. Caution!!! For arithmetical subsets of 2^ω, Σ^0_n classes are not the same as $\Sigma^{0,\emptyset^{(n-1)}}_1$ classes.

   (For instance, a Σ^0_2 class is the countable union of closed subsets of 2^ω, while a $\Sigma^{0,\emptyset'}_1$ class is an *open* subset of 2^ω.)

2. For $n \geq 1$, the following holds:

$$\text{weak } (n+1)\text{-randomness} \Rightarrow n\text{-randomness} \Rightarrow \text{weak } n\text{-randomness}$$

3. The reverse implications do not hold.

4. Effectively Typical Objects as Generic Objects

A possible worry

One might worry that the different choices of “generic properties” here are made in an ad hoc manner.

That is, one might question whether we have provided a principled response to the generic property problem.

Do the various effective notions of typicality as provide a particularly useful or informative account of arbitrary objects?

Effective analogues of set-theoretic notions

One line of response is to appeal to direct connections between effective notions of typicality and set-theoretic notions of genericity:

- weak n-genericity is an effective version of Cohen genericity;
- weak n-randomness is equivalent to Solovay n-genericity, which is an effective version of Solovay genericity.

To develop this response, one needs to argue (i) that set-theoretic notions of generic objects provide a reasonable account of arbitrary objects and (ii) that the "arbitrariness" of these set-theoretic generics is not lost in the passage to their effective analogues.

The typical Turing degree

A second line of response is to appeal to the role that these effective notions of typicality play in the study of the typical Turing degree. For example:

- Weakly 1-generic degrees are precisely the hyperimmune degrees (degrees that compute a function not dominated by any computable function).
- 2-randomness and 2-genericity feature prominently in these investigations.

While this is a more promising line to take, it is still vulnerable to the criticism that this is a rather narrow range of applicability.

Capturing typical behavior in classical mathematics

The strongest reason to take effective notions of typicality as providing a useful and informative account of arbitrary objects can be illustrated by the following examples:

Theorem (Bruckner, Leonard 1966)

A set $\mathcal{X} \subseteq [0,1]$ is the set of discontinuities of the derivative of some function $f : [0,1] \to \mathbb{R}$ if and only if $\mathcal{X}$ is a meager F_σ set.

Theorem (Kuyper, Terwijn 2013)

A real $x \in [0,1]$ is 1-generic if and only if for every computable differentiable $f : [0,1] \to \mathbb{R}$, f' is continuous at x.

Capturing typical behavior in classical mathematics (continued)

Theorem

A function $f : [0,1] \to \mathbb{R}$ of bounded variation is differentiable on a set of measure one.

Theorem (Brattka, Miller, Nies 2014)

A real $x \in [0,1]$ is 1-random if and only if every computable function $f : [0,1] \to \mathbb{R}$ of bounded variation is differentiable at x.

Theorem (Brattka, Miller, Nies 2014)

A real $x \in [0,1]$ is weakly 2-random if and only if every computable function $f : [0,1] \to \mathbb{R}$ that is differentiable on a set of measure one is differentiable at x.

Incomplete vs. complete specifications of typicality

The classical versions of these theorems tells us that certain properties are almost universal, but there is a small set of atypical exceptions.

As stated earlier, this yields an incomplete specification of the typical points.

By contrast, in the effective setting, we get a more complete specification of typical behavior.

That is, we have additional information concerning precisely where the typical behavior is guaranteed to occur.

Summing up

The account of arbitrary objects given by effective notions of typicality

- is rooted in the study of typical behavior given in terms of measure and category;
- allows us to analyze typical behavior that permits exceptions; and
- provides additional information about the exceptions to this typical behavior (information that is not available on the classical approach).

Thank you for your attention.