# Effectively closed classes, negligibility, and depth

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# Introduction

Let $\mathcal{P} \subseteq 2^\omega$ be an effectively closed subset of $2^\omega$ (also known as a $\Pi^0_1$ class).

Suppose we would like to produce a member of $\mathcal{P}$ by means of some combination of deterministic and probabilistic procedures.

More specifically, we want such a combination of procedures to produce a member of $\mathcal{P}$ with positive probability.

Our main question is:

- What obstacles could prevent us from succeeding?

# Introduction (continued)

Today I will discuss two such obstacles:

1. negligibility
2. depth

Negligible classes are precisely the classes whose members cannot be computed with positive probability by any combination of deterministic and probabilistic procedures.

Deep classes are even stronger: we cannot even produce an initial segment of some member of a deep class with sufficiently high probability (which I will make precise shortly).

# Outline of the talk

1. Background
2. Defining negligibility and depth
3. Basic results on negligible and deep classes
4. Examples

# 1. Background

# Martin-Löf randomness

## Definition

- A *Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly $\Sigma^0_1$ (i.e. effectively open) subsets of $2^\omega$ such that for each $i$,
$$\lambda(\mathcal{U}_i) \leq 2^{-i}.$$
- A sequence $X \in 2^\omega$ *passes the Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.
- $X \in 2^\omega$ is *Martin-Löf random*, denoted $X \in \mathrm{MLR}$, if $X$ passes every Martin-Löf test.

# Computable measures

## Definition

A measure $\mu$ on $2^\omega$ is *computable* if $\sigma \mapsto \mu(\llbracket\sigma\rrbracket)$ is computable as a real-valued function.

In other words, $\mu$ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that

$$|\mu(\llbracket\sigma\rrbracket) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

From now on we will write $\mu(\sigma)$ instead of $\mu(\llbracket\sigma\rrbracket)$.

# Examples of computable measures

- The Lebesgue measure $\lambda$
- For computable $p \in [0,1]$, the Bernoulli $p$-measure $\mu_p$, defined by
$$\mu_p(\sigma) = p^{\#_0(\sigma)}(1-p)^{\#_1(\sigma)},$$
where $\#_i(\sigma)$ denotes the number of $i$'s in $\sigma$ for $i = 0, 1$
- For a computable sequence $X \in 2^\omega$, the Dirac measure $\delta_X$ concentrated on $X$, defined by
$$\delta_X(\sigma) = \begin{cases} 1 & \text{if } \sigma \prec X \\ 0 & \text{if } \sigma \not\prec X \end{cases}$$

# Randomness with respect to computable measures

We will also be interested in sequences that are Martin-Löf random with respect to a computable measure.

## Definition

Let $\mu$ be a computable measure.

- A *$\mu$-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly $\Sigma^0_1$ subsets of $2^\omega$ such that for each $i$,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ is *$\mu$-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if $X$ passes every $\mu$-Martin-Löf test.

# Atomic computable measures

A measure $\mu$ is *atomic* if there is some $X \in 2^\omega$ such that $\mu(\{X\}) > 0$.

Note that if $X$ is an atom of a computable measure $\mu$, then $X \in \mathrm{MLR}_\mu$.

Every computable sequence is the atom of some computable measure, namely $\delta_X$.

In fact, the converse holds: if $X$ is the atom of a computable measure, then $X$ is a computable sequence.

# Computationally powerful random sequences

It is worth noting that some Martin-Löf random sequences can compute a member of every $\Pi^0_1$ class.

Recall that $X \in 2^\omega$ has *PA degree* if $X$ computes a consistent completion of Peano arithmetic.

An important result of Simpson's is that every sequence of PA degree computes a member of every $\Pi^0_1$ class.

Combining this with the fact that some Martin-Löf random sequences have PA degree yields the result.

# Stephan's dichotomy theorem

However, Stephan proved that this computational power is the exception and not the rule for Martin-Löf random sequences:

### Theorem (Stephan)

*If a Martin-Löf random has PA degree, it is already Turing complete (i.e., $A \geq_T \emptyset'$).*

Since the collection of sequences that compute $\emptyset'$ has Lebesgue measure zero, it follows that almost every Martin-Löf random sequence cannot compute a completion of PA.

# Difference randomness

This latter fact is related to a notion of randomness known as *difference randomness*.

## Definition

- A *difference test* is a computable sequence $((\mathcal{U}_i, \mathcal{V}_i))_{i\in\omega}$ of pairs of $\Sigma^0_1$ classes such that for each $i$,

$$\lambda(\mathcal{U}_i \setminus \mathcal{V}_i) \leq 2^{-i}.$$

- A sequence $X \in 2^\omega$ *passes the difference test* $((\mathcal{U}_i, \mathcal{V}_i))_{i\in\omega}$ if $X \notin \bigcap_i(\mathcal{U}_i \setminus \mathcal{V}_i)$.
- $X \in 2^\omega$ is *difference random* if $X$ passes every difference test.

# Difference randomness and Stephan's theorem

The following theorem is quite surprising:

**Theorem (Franklin, Ng)**

*Let $A$ be Martin-Löf random. Then $A$ is difference random if and only if $A \not\geq_T \emptyset'$.*

Combining this with Stephan's theorem yields:

**Corollary**

*Let $A$ be Martin-Löf random. Then $A$ is difference random if and only if $A$ does not have PA degree.*

# 2. Defining negligibility and depth

# A brief road-map

Negligibility and depth are both defined in terms of a certain measure that is in a certain sense universal.

To define this measure, we need to take a detour to discuss the following:

- left-c.e. semi-measures,
- universal semi-measures, and
- deriving a measure from a semi-measure.

Throughout this discussion, we will emphasize the connection to Turing functionals.

# Left-c.e. semi-measures

A *semi-measure* $\rho : 2^{<\omega} \to [0,1]$ satisfies

- $\rho(\varnothing) = 1$ and
- $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$ for every $\sigma \in 2^{<\omega}$.

We will be particularly interested in *left-c.e.* semi-measures.

A semi-measure $\rho$ is left-c.e. if each value $\rho(\sigma)$ is the limit of a non-decreasing computable sequence of rationals, uniformly in $\sigma$.

# Induced semi-measures

Recall: A *Turing functional* $\Phi : 2^\omega \to 2^\omega$ is given by a c.e. set of pairs of strings $(\sigma, \tau)$ such that if $(\sigma, \tau), (\sigma', \tau') \in \Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$.

For $\sigma \in 2^{<\omega}$, we define $\Phi^{-1}(\sigma) := \{X \in 2^\omega : \exists n\, (X{\restriction}n, \sigma) \in \Phi\}$.

## Proposition

(i) *If $\Phi$ is a Turing functional, then $\lambda_\Phi$, defined by*

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}(\sigma))$$

*for every $\sigma \in 2^{<\omega}$, is a left-c.e. semi-measure.*

(ii) *For every left c.e. semi-measure $\rho$, there is a Turing functional $\Phi$ such that $\rho = \lambda_\Phi$.*

# Universal semi-measures, 1

Levin proved the existence of a *universal* left-c.e. semi-measure.

A left-c.e. semi-measure $M$ is universal if for every left-c.e. semi-measure $\rho$, there is some $c \in \omega$ such that

$$\rho(\sigma) \leq c \cdot M(\sigma)$$

for every $\sigma \in 2^{<\omega}$.

# Universal semi-measures, 2

A universal semi-measures can be induced by a universal Turing functional.

For example, the functional $\Phi$ defined by

$$\Phi(1^e 0X) = \Phi_e(X)$$

is universal (where $(\Phi_e)_{e\in\omega}$ is an effective listing of all Turing functionals).

# The measure derived from a semi-measure

If $\rho$ is a semi-measure, we can define

$$\overline{\rho}(\sigma) := \inf_n \sum_{\tau \succeq \sigma \ \& \ |\tau|=n} \rho(\tau).$$

One can verify that $\overline{\rho}$ is the largest measure such that $\overline{\rho} \leq \rho$ (but it is not a probability measure in general).

## Proposition

*If $\rho$ is a left-c.e. semi-measure induced by a Turing functional $\Phi$, then*

$$\overline{\rho}(\sigma) = \lambda(\{X \in 2^\omega : X \in \Phi^{-1}(\sigma) \ \& \ \Phi(X) \text{ is total}\}).$$

# Negligible classes

Let $M$ be the universal left-c.e. semi-measure.

Then $\overline{M}$ can be seen as a universal measure (universal for all computable measures, as well as the measures derived from left-c.e. semi-measures).

**Definition**

$\mathcal{S} \subseteq 2^\omega$ is *negligible* if $\overline{M}(\mathcal{S}) = 0$.

# The intuition behind negligibility

Let $\mathcal{P}$ be a negligible $\Pi^0_1$ class.

$\overline{M}(\mathcal{P}) = 0$ means that the probability of producing some member of $\mathcal{P}$ by means of any Turing functional equipped with any sufficiently random oracle is 0.

To see this, note that

$$\overline{M}(\mathcal{P}) = 0 \text{ if and only if } \lambda\Big(\bigcup_{i\in\omega} \Phi_i^{-1}(\mathcal{P})\Big) = 0.$$

In particular, for each $\Phi_i$, $\lambda(\{X \in \mathrm{MLR} : \Phi_i(X) \in \mathcal{P}\}) = 0$.

# Deep classes: the idea

Depth is a property that is stronger than negligibility for $\Pi^0_1$ classes.

Instead of considering how difficult it is to produce a path through a $\Pi^0_1$ class $\mathcal{P}$, we can consider how difficult it is to produce an *initial segment* of some path through $\mathcal{P}$, level by level.

Deep classes are the "most difficult" of $\Pi^0_1$ classes in this respect.

# A few more definitions

Let $\mathcal{P} \subseteq 2^\omega$ be a $\Pi^0_1$ class.

Let $T^{ext} \subseteq 2^{<\omega}$ be the set of extendible nodes of $\mathcal{P}$,

$$T^{ext} = \{\sigma \in 2^{<\omega} : [\![\sigma]\!] \cap \mathcal{P} \neq \emptyset\}.$$

Thus $T^{ext}$ is the canonical co-c.e. tree such that $\mathcal{P} = [T^{ext}]$ (the set of infinite paths through $T^{ext}$).

For each $n \in \omega$, $T^{ext}_n$ consists of all strings in $T^{ext}$ of length $n$.

(I will write $T$ instead of $T^{ext}$ hereafter.)

# Deep classes: the definition

Let $\mathcal{P}$ be a $\Pi^0_1$ class and let $T$ be the canonical co-c.e. tree corresponding to $\mathcal{P}$.

$\mathcal{P}$ is a *deep class* if there is some computable, non-decreasing, unbounded function $h : \omega \to \omega$ such that

$$M(T_n) \leq 2^{-h(n)},$$

where $M(T_n) = \sum_{\sigma \in T_n} M(\sigma)$.

That is, the probability of producing some initial segment of a path through $\mathcal{P}$ is effectively bounded above.

# 3. Basic results on negligible and deep classes

# Members of negligible classes

A few observations:

- If a $\Pi^0_1$ class contains a computable member, clearly it cannot be negligible.
- Moreover, if a $\Pi^0_1$ class contains a Martin-Löf random member, it cannot be negligible, since any $\Pi^0_1$ class with a random member must have positive Lebesgue measure.

These two facts are subsumed by the following result:

## Proposition

*Let $\mathcal{P}$ be a negligible $\Pi^0_1$ class. Then for every computable measure $\mu$, $\mathcal{P}$ contains no $X \in \mathrm{MLR}_\mu$.*

# Does the converse hold?

Suppose that $\mathcal{P}$ is a $\Pi^0_1$ class such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure $\mu$.

Does it follow that $\mathcal{P}$ is negligible?

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# Does the converse hold?

Suppose that $\mathcal{P}$ is a $\Pi^0_1$ class such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure $\mu$.

Does it follow that $\mathcal{P}$ is negligible? No.

**Theorem (Bienvenu, Porter, Taveneaux)**

*There is a non-negligible $\Pi^0_1$ class $\mathcal{P}$ such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure $\mu$.*

# The main ingredients of the proof

- A $\Pi^0_1$ class $\mathcal{P}$ is thin if for every $\Pi^0_1$ subclass $\mathcal{Q} \subseteq \mathcal{P}$, there is some $\sigma \in 2^{<\omega}$ such that $\mathcal{Q} = \mathcal{P} \cap [\![\sigma]\!]$.
- If a thin class $\mathcal{P}$ has a computable member $X$, then $X$ is isolated in $\mathcal{P}$ (since $\{X\}$ is a $\Pi^0_1$ subclass of $\mathcal{P}$).
- Downey, Greenberg, and Miller have constructed a non-negligible, perfect thin $\Pi^0_1$ class $\mathcal{P}$.
- By the second point above, a perfect thin class cannot have any computable paths, and hence $\mathcal{P}$ does not contain any atoms of any computable atomic measure.
- Simpson showed that every thin class has Lebesgue measure 0; using the previous fact, we show Simpson's result holds for every computable measure.

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**Theorem (Bienvenu, Porter, Taveneaux)**

*There is a negligible class $\mathcal{P}$ that is not deep.*

We use a finite injury argument to keep the measure of $\mathcal{P}$ sufficiently high at each finite level while ensuring that this measure eventually converges to 0.

# Why use the co-c.e. tree in the definition of depth?

For every $\Pi^0_1$ class $\mathcal{P}$ there is a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{P} = [T]$.

Why can't we use this computable tree $T$ in the definition of depth?

In general, $T$ will contain non-extendible nodes, so even if we can compute some element in $T_n$, we still may fail to compute an initial segment of a member of $\mathcal{P}$.

Can we give a better reason to restrict our attention to the canonical co-c.e. tree?

# Vindicating the definition of depth

**Theorem (Bienvenu, Porter, Taveneaux)**

*Let $T$ be a computable tree. Then there is no computable order $h$ such that $M(T_n) \leq 2^{-h(n)}$ for every $n \in \omega$.*

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Suppose $M(T_n) \leq 2^{-h(n)}$ for some computable order $h$.

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Suppose $M(T_n) \leq 2^{-h(n)}$ for some computable order $h$.

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**Proof.**

Suppose $M(T_n) \leq 2^{-h(n)}$ for some computable order $h$.

Case 1: $T$ has only finitely many non-extendible nodes.

Then the leftmost path $X$ of $T$ is computable (since $T$ is computable).

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We define a semi-measure $\rho$ such that $\rho(\sigma_n) = 2^{-K(n)}$ for every $n$ (consistently extending $\rho$ to initial segments of each $\sigma_n$), where $K(n)$ is the prefix-free Kolmogorov complexity of $n$.

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But then by our assumption, $2^{-h(f(n))} \geq M(T_{f(n)}) \geq 2^{-K(n)-c}$, and hence $h(f(n)) \leq K(n) + c$.

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Then there is some $c$ such that

$$M(T_{f(n)}) \geq 2^{-c}\rho(T_{f(n)}) \geq 2^{-K(n)-c}.$$

But then by our assumption, $2^{-h(f(n))} \geq M(T_{f(n)}) \geq 2^{-K(n)-c}$, and hence $h(f(n)) \leq K(n) + c$.

This contradicts the fact that there is no computable lower bound for $K$.

# Randoms computing members of deep classes

Which Martin-Löf random sequences can compute some member of a deep class?

We've already seen that if a Martin-Löf random sequence $X$ has PA degree, it can compute a member of every deep class.

Thus, by Stephan's dichotomy theorem, if $X \in \mathrm{MLR}$ and $X \geq_T \emptyset'$, $X$ computes some member of a deep class.

But this is the best we can do.

**Theorem (Bienvenu, Porter, Taveneaux)**

*No difference random sequence can compute a member of a deep class.*

# 4. Examples

# Examples of deep classes

There are a number of deep classes that naturally arise in computability theory.

We don't, however, have any "natural" examples of negligible classes that aren't deep.

I will briefly discuss three examples of deep classes:

1. consistent completions of Peano arithmetic;
2. shift-complex sequences; and
3. $h$-diagonally non-computable functions.

# Consistent completions of Peano arithmetic

The following is implicit in work of Levin and Stephan.

**Theorem**

*The $\Pi^0_1$ class of consistent completions of PA is a deep class.*

The idea is to define a partial computable $\{0, 1\}$-valued function $f$ (using the recursion theorem) in such a way that we diagonalize against large classes of oracles that could potentially compute a total extension of $f$.

# Shift-complex sequences: the idea

Although a Martin-Löf random sequence $X$ has high initial segment complexity, satisfying

$$K(X{\upharpoonright}n) \geq n - O(1),$$

$X$ will still contain arbitrarily long runs of 0s (since all Martin-Löf random sequences are normal).

That is, certain subwords of $X$ can have fairly low initial segment complexity.

By contrast, a shift-complex sequence is a sequence with the property that every subword has high initial segment complexity.

# Shift-complex sequences: the formal definition

For $\delta \in (0,1)$ and $c \in \omega$, we say that $X \in 2^\omega$ is *$(\delta, c)$-shift complex* if

$$K(\tau) \geq \delta|\tau| - c$$

for every subword $\tau$ of $X$.

The following draws upon work of Rumyantsev.

Theorem (Bienvenu, Porter, Taveneaux)

*For every $\delta \in (0,1)$ and $c \in \omega$, the $(\delta, c)$-shift complex sequences form a deep class.*

# Diagonally non-computable sequences and randomness

Recall that a sequence $X$ is diagonally non-computable if there is some total function $f \leq_T X$ such that $f(e) \neq \phi_e(e)$ for every $e$.

Every Martin-Löf random sequence $X$ is diagonally non-computable:

Let $f(e) = X{\upharpoonright}e$ (coded as a natural number).

Note that $f(e) < 2^{e+1}$.

# $\mathrm{DNC}_h$ functions

Let $h$ be a computable, non-decreasing, unbounded function.

$f$ is a $\mathrm{DNC}_h$ function if

- $f$ is total,
- $f(e) \neq \phi_e(e)$ for every $e$, and
- $f(e) < h(e)$ for every $e$.

Theorem (Bienvenu, Porter, Taveneaux)

*$DNR_h$ is a deep class if and only if* $\sum_{n=0}^{\infty} \frac{1}{h(n)} = \infty$.

Moreover, if $\sum_{n=0}^{\infty} \frac{1}{h(n)} < \infty$, then every Martin-Löf random computes a $\mathrm{DNC}_h$ function.