Effectively closed classes, negligibility, and depth

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Hauptseminar Mathematische Logik und Theoretische Informatik
Universität Heidelberg
19 November 2013

Introduction

Let $\mathcal{P} \subseteq 2^\omega$ be an effectively closed subset of 2^ω (i.e. a Π^0_1 class).

Suppose we would like to produce a member of $\mathcal{P}$ by means of some combination of deterministic and probabilistic procedures.

More specifically, we want a combination of these procedures to produce a member of $\mathcal{P}$ with positive probability.

Our main question is:

- What obstacles could prevent us from succeeding?

Introduction (continued)

Today I will discuss two such obstacles:

1. negligibility
2. depth

Negligible classes are precisely the classes whose members cannot be computed with positive probability by any combination of deterministic and probabilistic procedures.

Deep classes are even stronger: we cannot even produce an initial segment of some member of a deep class with sufficiently high probability (which I will make precise shortly).

Outline of the talk

1. Background
2. Defining negligibility and depth
3. Basic results on negligible and deep classes
4. Examples

1. Background

Martin-Löf randomness

Definition

- A *Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i \in \omega}$ of uniformly Σ^0_1 (i.e. effectively open) subsets of 2^ω such that for each i,

$$\lambda(\mathcal{U}_i) \leq 2^{-i}.$$

- A sequence $X \in 2^\omega$ *passes the Martin-Löf test* $(\mathcal{U}_i)_{i \in \omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

- $X \in 2^\omega$ is *Martin-Löf random*, denoted $X \in \mathrm{MLR}$, if X passes every Martin-Löf test.

Computable measures

Definition

A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu(\llbracket\sigma\rrbracket)$ is computable as a real-valued function.

In other words, μ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that

$$|\mu(\llbracket\sigma\rrbracket) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

From now on we will write $\mu(\sigma)$ instead of $\mu(\llbracket\sigma\rrbracket)$.

Randomness with respect to computable measures

We will also be interested in sequences that are Martin-Löf random with respect to a computable measure.

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly Σ^0_1 subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes every μ-Martin-Löf test.

Atomic computable measures

A measure μ is *atomic* if there is some $X \in 2^\omega$ such that $\mu(\{X\}) > 0$.

- Note that if X is an atom of a computable measure μ, then $X \in \mathsf{MLR}_\mu$.
- Every computable sequence is the atom of some computable measure, namely, the Dirac measure δ_X concentrated on X, defined by
$$\delta_X(\sigma) = \begin{cases} 1 & \text{if } \sigma \prec X \\ 0 & \text{if } \sigma \not\prec X \end{cases}.$$
- In fact, the converse holds: if X is the atom of a computable measure, then X is a computable sequence.

Computationally powerful random sequences

It is worth noting that some Martin-Löf random sequences can compute a member of every Π^0_1 class.

Recall that $X \in 2^\omega$ has *PA degree* if X computes a consistent completion of Peano arithmetic.

It is well known that every sequence of PA degree computes a member of every Π^0_1 class.

Combining this with the fact that some Martin-Löf random sequences have PA degree yields the result.

Stephan's dichotomy theorem

However, Stephan proved that this computational power is the exception and not the rule for Martin-Löf random sequences:

Theorem (Stephan)

If a Martin-Löf random sequence A has PA degree, it is already Turing complete (i.e., $A \geq_T \emptyset'$).

Since the collection of sequences that compute $\emptyset'$ has Lebesgue measure zero, it follows that almost every Martin-Löf random sequence cannot compute a completion of PA.

Difference randomness

This latter fact is related to a notion of randomness known as *difference randomness*.

Definition

- A *difference test* is a computable sequence $((\mathcal{U}_i, \mathcal{V}_i))_{i\in\omega}$ of pairs of Σ^0_1 classes such that for each i,
$$\lambda(\mathcal{U}_i \setminus \mathcal{V}_i) \leq 2^{-i}.$$
- A sequence $X \in 2^\omega$ *passes the difference test* $((\mathcal{U}_i, \mathcal{V}_i))_{i\in\omega}$ if $X \notin \bigcap_i (\mathcal{U}_i \setminus \mathcal{V}_i)$.
- $X \in 2^\omega$ is *difference random* if X passes every difference test.

Difference randomness and Stephan's theorem

The following theorem is quite surprising:

Theorem (Franklin, Ng)

Let A be Martin-Löf random. Then A is difference random if and only if $A \not\geq_T \emptyset'$.

Combining this with Stephan's theorem yields:

Corollary

Let A be Martin-Löf random. Then A is difference random if and only if A does not have PA degree.

2. Defining negligibility and depth

A brief road-map

Negligibility is defined in terms of a measure that is in a certain sense universal.

To define this measure, we need to take a detour to discuss the following:

- left-c.e. semi-measures,
- universal semi-measures, and
- deriving a measure from a semi-measure.

Throughout this discussion, we will emphasize the connection to Turing functionals.

Left-c.e. semi-measures

A *semi-measure* $\rho : 2^{<\omega} \to [0,1]$ satisfies

- $\rho(\varnothing) = 1$ and
- $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$ for every $\sigma \in 2^{<\omega}$.

We will be particularly interested in *left-c.e.* semi-measures.

A semi-measure ρ is left-c.e. if each value $\rho(\sigma)$ is the limit of a non-decreasing computable sequence of rationals, uniformly in σ.

Induced semi-measures

Recall: A *Turing functional* $\Phi : 2^\omega \to 2^\omega$ is given by a c.e. set of pairs of strings (σ, τ) such that if $(\sigma, \tau), (\sigma', \tau') \in \Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$.

For $\sigma \in 2^{<\omega}$, we define $\Phi^{-1}(\sigma) := \{X \in 2^\omega : \exists n\, (X{\upharpoonright}n, \sigma) \in \Phi\}$.

Proposition

(i) *If Φ is a Turing functional, then λ_Φ, defined by*

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}(\sigma))$$

for every $\sigma \in 2^{<\omega}$, is a left-c.e. semi-measure.

(ii) *For every left c.e. semi-measure ρ, there is a Turing functional Φ such that $\rho = \lambda_\Phi$.*

Universal semi-measures, 1

Levin proved the existence of a *universal* left-c.e. semi-measure.

A left-c.e. semi-measure M is universal if for every left-c.e. semi-measure ρ, there is some $c \in \omega$ such that

$$\rho(\sigma) \leq c \cdot M(\sigma)$$

for every $\sigma \in 2^{<\omega}$.

Universal semi-measures, 2

A universal semi-measures can be induced by a universal Turing functional.

For example, the functional Φ defined by

$$\Phi(1^e0X) = \Phi_e(X)$$

is universal (where $(\Phi_e)_{e\in\omega}$ is an effective listing of all Turing functionals).

The measure derived from a semi-measure

If ρ is a semi-measure, we can define

$$\overline{\rho}(\sigma) := \inf_n \sum_{\tau \succeq \sigma \ \&\ |\tau|=n} \rho(\tau).$$

One can verify that $\overline{\rho}$ is the largest measure such that $\overline{\rho} \leq \rho$ (but it is not a probability measure in general).

Proposition

If ρ is a left-c.e. semi-measure induced by a Turing functional Φ, then

$$\overline{\rho}(\sigma) = \lambda(\{X \in 2^\omega : X \in \Phi^{-1}(\sigma) \ \&\ \Phi(X) \text{ is total}\}).$$

Negligible classes

Let M be the universal left-c.e. semi-measure.

Then $\overline{M}$ can be seen as a universal measure (universal for all computable measures, as well as the measures derived from left-c.e. semi-measures).

Definition

$\mathcal{S} \subseteq 2^\omega$ is *negligible* if $\overline{M}(\mathcal{S}) = 0$.

The intuition behind negligibility

Let $\mathcal{P}$ be a negligible Π^0_1 class.

$\overline{M}(\mathcal{P}) = 0$ means that the probability of producing some member of $\mathcal{P}$ by means of any Turing functional equipped with any sufficiently random oracle is 0.

To see this, note that

$$\overline{M}(\mathcal{P}) = 0 \text{ if and only if } \lambda\Big(\bigcup_{i\in\omega} \Phi_i^{-1}(\mathcal{P})\Big) = 0.$$

In particular, for each Φ_i, $\lambda(\{X \in \mathrm{MLR} : \Phi_i(X) \in \mathcal{P}\}) = 0$.

Deep classes: the idea

Depth is a property that is stronger than negligibility for Π^0_1 classes.

Instead of considering how difficult it is to produce a path through a Π^0_1 class $\mathcal{P}$, we can consider how difficult it is to produce an *initial segment* of some path through $\mathcal{P}$, level by level.

Deep classes are the "most difficult" of Π^0_1 classes in this respect.

A few more definitions

Let $\mathcal{P} \subseteq 2^\omega$ be a Π^0_1 class.

Let $T^{ext} \subseteq 2^{<\omega}$ be the set of extendible nodes of $\mathcal{P}$,

$$T^{ext} = \{\sigma \in 2^{<\omega} : [\![\sigma]\!] \cap \mathcal{P} \neq \emptyset\}.$$

Thus T^{ext} is the canonical co-c.e. tree such that $\mathcal{P} = [T^{ext}]$ (the set of infinite paths through T^{ext}).

For each $n \in \omega$, T^{ext}_n consists of all strings in T^{ext} of length n.

(I will write T instead of T^{ext} hereafter.)

Deep classes: the definition

Let $\mathcal{P}$ be a Π^0_1 class and let T be the canonical co-c.e. tree such that $\mathcal{P} = [T]$.

$\mathcal{P}$ is a *deep class* if there is some computable, non-decreasing, unbounded function $h : \omega \to \omega$ such that

$$M(T_n) \leq 2^{-h(n)},$$

where $M(T_n) = \sum_{\sigma \in T_n} M(\sigma)$.

That is, the probability of producing some initial segment of a path through $\mathcal{P}$ is effectively bounded above.

3. Basic results on negligible and deep classes

Members of negligible classes

A few observations:

- If a Π^0_1 class contains a computable member, clearly it cannot be negligible.
- Moreover, if a Π^0_1 class contains a Martin-Löf random member, it cannot be negligible, since any Π^0_1 class with a random member must have positive Lebesgue measure.

These two facts are subsumed by the following result:

Proposition

Let $\mathcal{P}$ be a negligible Π^0_1 class. Then for every computable measure μ, $\mathcal{P}$ contains no $X \in \mathrm{MLR}_\mu$.

Does the converse hold?

Suppose that $\mathcal{P}$ is a Π^0_1 class such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure μ.

Does it follow that $\mathcal{P}$ is negligible?

Does the converse hold?

Suppose that $\mathcal{P}$ is a Π^0_1 class such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure μ.

Does it follow that $\mathcal{P}$ is negligible? No.

Does the converse hold?

Suppose that $\mathcal{P}$ is a Π^0_1 class such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure μ.

Does it follow that $\mathcal{P}$ is negligible? No.

Theorem (Bienvenu, Porter, Taveneaux)

There is a non-negligible Π^0_1 class $\mathcal{P}$ such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure μ.

The main ingredients of the proof

- A Π^0_1 class $\mathcal{P}$ is thin if for every Π^0_1 subclass $\mathcal{Q} \subseteq \mathcal{P}$, there is some $\sigma \in 2^{<\omega}$ such that $\mathcal{Q} = \mathcal{P} \cap [\![\sigma]\!]$.
- If a thin class $\mathcal{P}$ has a computable member X, then X is isolated in $\mathcal{P}$ (since $\{X\}$ is a Π^0_1 subclass of $\mathcal{P}$).
- Downey, Greenberg, and Miller have constructed a non-negligible, perfect thin Π^0_1 class $\mathcal{P}$.
- By the second point above, a perfect thin class cannot have any computable paths, and hence $\mathcal{P}$ does not contain any atoms of any computable atomic measure.
- Simpson showed that every thin class has Lebesgue measure 0; using the previous fact, we show Simpson's result holds for every computable measure.

Depth vs. negligibility

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Theorem (Bienvenu, Porter, Taveneaux)

There is a negligible class $\mathcal{P}$ that is not deep.

We use a finite injury argument to keep the measure of $\mathcal{P}$ sufficiently high at each finite level while ensuring that this measure eventually converges to 0.

Why use the co-c.e. tree in the definition of depth?

For every Π^0_1 class $\mathcal{P}$ there is a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{P} = [T]$.

Why can't we use this computable tree T in the definition of depth?

In general, T will contain non-extendible nodes, so even if we can compute some element in T_n, we still may fail to compute an initial segment of a member of $\mathcal{P}$.

Can we give a better reason to restrict our attention to the canonical co-c.e. tree?

Vindicating the definition of depth

Theorem (Bienvenu, Porter, Taveneaux)

Let T be a computable tree. Then there is no computable order h such that $M(T_n) \leq 2^{-h(n)}$ for every $n \in \omega$.

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Suppose $M(T_n) \leq 2^{-h(n)}$ for some computable order h.

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Proof.

Suppose $M(T_n) \leq 2^{-h(n)}$ for some computable order h.

Case 1: T has only finitely many non-extendible nodes.

Then the leftmost path X of T is computable (since T is computable).

Vindicating the definition of depth, 2

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But then by our assumption, $2^{-h(f(n))} \geq M(T_{f(n)})$

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Then there is some c such that

$$M(T_{f(n)}) \geq 2^{-c}\rho(T_{f(n)}) \geq 2^{-K(n)-c}.$$

But then by our assumption, $2^{-h(f(n))} \geq M(T_{f(n)}) \geq 2^{-K(n)-c}$, and hence $h(f(n)) \leq K(n) + c$.

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Then there is some c such that

$$M(T_{f(n)}) \geq 2^{-c}\rho(T_{f(n)}) \geq 2^{-K(n)-c}.$$

But then by our assumption, $2^{-h(f(n))} \geq M(T_{f(n)}) \geq 2^{-K(n)-c}$, and hence $h(f(n)) \leq K(n) + c$.

This contradicts the fact that there is no computable lower bound for K.

Randoms computing members of deep classes

Which Martin-Löf random sequences can compute some member of a deep class?

We've already seen that if a Martin-Löf random sequence X has PA degree, it can compute a member of every deep class.

Thus, by Stephan's dichotomy theorem, if $X \in \mathrm{MLR}$ and $X \geq_T \emptyset'$, X computes some member of a deep class.

But this is the best we can do.

Theorem (Bienvenu, Porter, Taveneaux)

No difference random sequence can compute a member of a deep class.

4. Examples

Examples of deep classes

There are a number of deep classes that naturally arise in computability theory.

We don't, however, have any "natural" examples of negligible classes that aren't deep.

I will briefly discuss three examples of deep classes:

1. consistent completions of Peano arithmetic;
2. shift-complex sequences; and
3. h-diagonally non-computable functions.

Consistent completions of Peano arithmetic

The following is implicit in work of Levin and Stephan.

Theorem

The Π^0_1 class of consistent completions of PA is a deep class.

The idea is to define a partial computable $\{0,1\}$-valued function f (using the recursion theorem) in such a way that we diagonalize against large classes of oracles that could potentially compute a total extension of f.

Shift-complex sequences: the idea

Although a Martin-Löf random sequence X has high initial segment complexity, satisfying

$$K(X{\upharpoonright}n) \geq n - O(1),$$

X will still contain arbitrarily long runs of 0s (since all Martin-Löf random sequences are normal).

That is, certain subwords of X can have fairly low initial segment complexity.

By contrast, a shift-complex sequence is a sequence with the property that every subword has high initial segment complexity.

Shift-complex sequences: the formal definition

For $\delta \in (0,1)$ and $c \in \omega$, we say that $X \in 2^\omega$ is *(δ, c)-shift complex* if

$$K(\tau) \geq \delta|\tau| - c$$

for every subword τ of X.

The following draws upon work of Rumyantsev.

Theorem (Bienvenu, Porter, Taveneaux)

For every $\delta \in (0,1)$ and $c \in \omega$, the (δ, c)-shift complex sequences form a deep class.

Diagonally non-computable sequences and randomness

Recall that a sequence X is diagonally non-computable if there is some total function $f \leq_T X$ such that $f(e) \neq \phi_e(e)$ for every e.

Every Martin-Löf random sequence X is diagonally non-computable:

Let $f(e) = X \upharpoonright e$ (coded as a natural number).

Note that $f(e) < 2^{e+1}$.

DNC$_h$ functions

Let h be a computable, non-decreasing, unbounded function.

f is a DNC$_h$ function if

- f is total,
- $f(e) \neq \phi_e(e)$ for every e, and
- $f(e) < h(e)$ for every e.

Theorem (Bienvenu, Porter, Taveneaux)

DNC$_h$ is a deep class if and only if $\sum_{n=0}^{\infty} \frac{1}{h(n)} = \infty$.

Moreover, if $\sum_{n=0}^{\infty} \frac{1}{h(n)} < \infty$, then every Martin-Löf random computes a DNC$_h$ function.

Thank you for your attention!