Randomness and Semi-Measures

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Journeés Calculabilités 2013
Nancy
12 April 2013

Motivating the Problem

Algorithmic randomness with respect to a measure is fairly well understood, for both computable and non-computable measures.

In this talk, I will discuss recent joint work with Laurent Bienvenu, Rupert Hölzl, and Paul Shafer on finding a natural and useful definition of randomness with respect to a semi-measure.

In particular, we will focus on randomness with respect to a left-c.e. (or lower semi-computable) semi-measure.

Outline

1. Randomness with respect to a measure

Some notation

$2^{<\omega}$ is the collection of finite binary sequences.

2^{ω} is the collection of infinite binary sequences.

The standard topology on 2^{ω} is given by the basic open sets

$$[\![\sigma]\!] = \{X \in 2^{\omega} : \sigma \prec X\},$$

where $\sigma \in 2^{<\omega}$ and $\sigma \prec X$ means that σ is an initial segment of X.

Lastly, the Lebesgue measure on 2^{ω}, denoted λ, is defined by

$$\lambda([\![\sigma]\!]) = 2^{-|\sigma|}$$

for each $\sigma \in 2^{<\omega}$ (where $|\sigma|$ is the length of σ), and then we extend λ to all Borel sets in the usual way.

Computable probability measures on 2^ω

A probability measure μ on 2^ω is *computable* if $\sigma \mapsto \mu([\![\sigma]\!])$ is computable as a real-valued function, i.e., if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that

$$|\mu([\![\sigma]\!]) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

From now on, we will write $\mu([\![\sigma]\!])$ as $\mu(\sigma)$.

We've already seen one example of a computable measure: the Lebesgue measure.

MLR with respect to a computable measure

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 (i.e. effectively open) subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- A sequence $X \in 2^\omega$ *passes the μ-Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.
- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes *every* μ-Martin-Löf test.

Turing functionals

Recall: A *Turing functional* $\Phi : 2^\omega \to 2^\omega$ is a c.e. set of pairs of strings (σ, τ) such that if $(\sigma, \tau), (\sigma', \tau') \in \Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$.

Given $\sigma \in 2^\omega$, $\Phi^\sigma := \bigcup\{\tau : \exists \sigma' \preceq \sigma(\sigma', \tau) \in \Phi\}$.

Further, given $B \in 2^\omega$, $\Phi(B) := \bigcup_n \Phi^{B\restriction n}$.

Equivalently, $\Phi(B) = \bigcup\{\tau : \exists n(B\restriction n, \tau) \in \Phi\}$.

If $\Phi(B) \in 2^\omega$, we say $\Phi(B)$ is defined, denoted $\Phi(B)\downarrow$.

A Turing functional Φ is *almost total* if

$$\lambda(\mathrm{dom}(\Phi)) = 1.$$

Computable measures and Turing functionals

Given an almost total Turing functional Φ, the *measure induced by* Φ, denoted λ_Φ, is defined by

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}(\sigma)) = \lambda(\{X : \Phi^X \succ \sigma\})$$

It's not hard to verify that λ_Φ is a computable measure.

Moreover, given a computable measure μ, there is some almost total functional Φ such that $\mu = \lambda_\Phi$.

Preservation of randomness

The following result is very useful.

Theorem

Given Φ *is an almost total Turing functional and* $X \in \mathrm{MLR}$, $\Phi(X) \in \mathrm{MLR}_{\lambda_\Phi}$.

Non-computable measures on 2^ω

Let $\mathcal{P}(2^\omega)$ be the collection of probability measures on 2^ω.

To define randomness for a non-computable measure, we need to have access to the measure in some way.

In order to have access to the measure, we need to code it as a sequence, which we will use as an oracle in defining our tests.

We will fix such a coding map $\Theta : 2^\omega \to \mathcal{P}(2^\omega)$ (the details of which we won't consider here).

Given a measure μ, if $\Theta(M) = \mu$, we will refer to M as a *representation of μ*.

Θ is defined in such a way that each measure has many representations.

MLR with respect to a non-computable measure, 1

Definition

Let μ be a non-computable measure, and let M be a representation of μ.

- An *M-Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of $\Sigma^0_1(M)$ (i.e. M-effectively open) subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ is *M-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu^M$, if X passes *every* M-Martin-Löf test.

MLR with respect to a non-computable measure, 2

Definition

Let μ be a non-computable measure.

$X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if there is some representation M of μ such that X is M-Martin-Löf random.

Blind randomness

An alternative approach to defining randomness with respect to a non-computable measure dispenses with the representations.

Definition

Let μ be a non-computable measure.

- A *blind μ-Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 (i.e. effectively open) subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ is *blind μ-Martin-Löf random*, denoted $X \in \mathrm{bMLR}_\mu$, if X passes *every* blind μ-Martin-Löf test.

2. Left-c.e. semi-measures

What is a semi-measure?

A semi-measure can be seen as a defective probability measure.

Whereas a probability measure μ on 2^ω satisfies

- $\mu(\varnothing) = 1$ and
- $\mu(\sigma) = \mu(\sigma 0) + \mu(\sigma 1)$ for every $\sigma \in 2^{<\omega}$,

a semi-measure ρ on 2^ω satisfies

- $\rho(\varnothing) \leq 1$ and
- $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$ for every $\sigma \in 2^{<\omega}$.

Given that every probability measure on 2^ω is a semi-measure on 2^ω, it's not unreasonable to seek to extend the definition of randomness with respect to a measure to a definition of randomness with respect to a semi-measure.

Left-c.e. semi-measures

Henceforth, we will restrict our attention to the class of left-c.e. semi-measures.

A semi-measure ρ is *left-c.e.* (or lower semi-computable) if, uniformly in σ, there is a computable non-decreasing sequence $(q_i)_{i\in\omega}$ such that

$$\lim_{i\to\infty} q_i = \rho(\sigma).$$

That is, the values of ρ on basic open sets are uniformly approximable from below.

Why restrict to left-c.e. semi-measures?

The answer is: left-c.e. semi-measures are precisely the class of semi-measures that are induced by Turing functionals.

That is, for every Turing functional Φ, the function

$$\lambda_{\Phi}(\sigma) = \lambda(\Phi^{-1}(\sigma)) = \lambda(\{X : \Phi^X \succ \sigma\})$$

is a left-c.e. semi-measure.

Moreover, for every left-c.e. semi-measure ρ, there is a Turing functional Φ such that $\rho = \lambda_{\Phi}$.

Conditions for a definition of randomness

What conditions do we want a definition of randomness with respect to a semi-measure to satisfy?

First, we want it to extend the definition of randomness with respect to a measure:

- If X is random with respect to a measure μ, we also want X to be random with respect to μ considered as a semi-measure.

Second, it'd be nice to have a version of the preservation of randomness theorem:

- If X is random and Φ is a Turing functional, then $\Phi(X)$ is random with respect to the semi-measure λ_Φ.

A first approach to randomness wrt a semi-measure

Why not simply replace the measure μ in the definition of μ-Martin-Löf randomness with a left-c.e. semi-measure ρ?

Let's say a *ρ-test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 subsets of 2^ω such that for each i,

$$\rho(\mathcal{U}_i) \leq 2^{-i}.$$

Can we define randomness with respect to a semi-measure in terms of ρ-tests?

The drawback of ρ-tests

Unfortunately, ρ-tests don't behave so nicely:

Proposition (BHPS)

There is a left-c.e. semi-measure ρ such that for any uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 subsets of 2^ω satisfying, for every $i \in \omega$,

$$\rho(\mathcal{U}_i) \leq 2^{-i},$$

we have $\bigcap_{i\in\omega} \mathcal{U}_i = \emptyset$.

Thus, if we were to count a sequence as Martin-Löf random with respect to a semi-measure ρ if it avoids all ρ-tests, then every sequence would be random with respect to the above-mentioned semi-measure.

A second approach to randomness wrt a semi-measure

Recently, Shen asked the following question.

Question

If Φ and Ψ are Turing functionals that induce the same semi-measure, i.e.,

$$\lambda_\Phi = \lambda_\Psi,$$

does it follow that $\Phi(\mathrm{MLR}) = \Psi(\mathrm{MLR})$?

A positive answer to Shen's question might justify the following definition:

Y is random with respect to a semi-measure ρ if for any Turing functional Φ such that $\rho = \lambda_\Phi$, there is some $X \in \mathrm{MLR}$ such that $\Phi(X) = Y$.

A negative answer to Shen's question

But we have the following.

Proposition (BHPS)

There exist Turing functionals Φ *and* Ψ *such that*

$$\lambda_\Phi = \lambda_\Psi$$

and

$$\Phi(\mathsf{MLR}) \neq \Psi(\mathsf{MLR}).$$

Proof idea

Consider Chaitin's Ω, a nicely approximable Martin-Löf random sequence.

We can define a Turing functional Φ such that $\mathrm{dom}(\Phi) = \{\Omega\}$ and $\Phi(\Omega) = 0^\omega$.

Using the definition of Φ as a blueprint, we can define a functional Ψ that maps the same amount of measure to each string, but which satisfies $\mathrm{dom}(\Psi) = \{0^\omega\}$ and $\Psi(0^\omega) = 0^\omega$.

Thus $\Phi(\mathrm{MLR}) = \{0^\omega\}$ and $\Psi(\mathrm{MLR}) = \emptyset$.

3. Restricting semi-measures to measures

A semi-measure as a network flow

It is helpful to think of a semi-measure as a network flow through the full binary tree:

We initially give the node at the root of the tree some amount of flow ≤ 1 ($\rho(\varnothing) \leq 1$).

Some amount of this flow at each node σ is passed along to the node corresponding to $\sigma 0$, some is passed along to the node corresponding to $\sigma 1$, and potentially, some of the flow is lost. ($\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$).

The bar of a semi-measure

Using this idea, we can define the largest measure less than a given semi-measure.

The idea is to ignore all of the flow that is lost from the network, so that for a given node, we consider the amount of flow that passes through it and is never lost.

$$\overline{\rho}(\sigma) := \inf_n \sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau)$$

One can verify that $\overline{\rho}$ is the largest measure such that $\overline{\rho} \leq \rho$ (but it is not a probability measure in general).

The bar of a semi-measure and Turing functionals

One particularly nice feature of $\overline{\rho}$ is its connection to Turing functionals.

If

$$\rho(\sigma) = \lambda(\{X : \Phi^X \succ \sigma\}),$$

then

$$\overline{\rho}(\sigma) = \lambda(\{X : \Phi(X)\downarrow \;\&\; \Phi^X \succ \sigma\}).$$

Two more candidate definitions

1. Define $\mathrm{MLR}_\rho := \{X : X \in \mathrm{MLR}_{\overline{\rho}}\}$
2. Define $\mathrm{MLR}_\rho := \{X : X \in \mathrm{bMLR}_{\overline{\rho}}\}$

Why consider option 2 as opposed to option 1?

Because $\overline{\rho}$ can encode lots of information.

Encoding information in $\overline{\rho}$

Theorem (BHPS)

There is a left-c.e. semi-measure ρ and some $\alpha \in (0,1)$ such that

- $\overline{\rho} = \alpha \cdot \lambda$; *and*
- $\alpha \equiv_T \emptyset''$.

There are two ways to "control" the value $\overline{\rho}(\sigma)$:

1. Increase the value of the current approximation of $\rho(\sigma)$.
2. Increase the amount of flow the leaves the network below σ.

Some consequences

Given the ρ from the previous theorem, any representation of $\overline{\rho}$ must compute $\emptyset''$.

Thus if M is a representation of $\overline{\rho}$,

$$X \in \mathrm{MLR}_{\overline{\rho}}^{M} \;\Rightarrow\; X \text{ is at least 3 - random.}$$

However,

$$X \in \mathrm{bMLR}_{\overline{\rho}} \;\Leftrightarrow\; X \in \mathrm{MLR},$$

since every blind $\overline{\rho}$-test is simply a Martin-Löf test, and vice versa.

No preservation of randomness

There is still a problem:

Proposition (BHPS)

There is a semi-measure ρ such that

- $\rho = \lambda_\Phi$ *for some Turing functional* Φ*;*
- $\mathrm{dom}(\Phi) \cap \mathrm{MLR} \neq \emptyset$*; and*
- $\mathrm{bMLR}_{\overline{\rho}} = \emptyset$.

That is, preservation of randomness fails in this case.

4. Weak 2-randomness and semi-measures

Weak 2-randomness

Definition

Let μ be a computable measure.

- A *generalized μ-Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 (i.e. effectively open) subsets of 2^ω such that

$$\lim_{i\to\infty} \mu(\mathcal{U}_i) = 0.$$

- $X \in 2^\omega$ is *μ-weakly 2-random*, denoted $X \in \mathrm{W2R}_\mu$, if X passes *every* μ-Martin-Löf test.

We can also define weak 2-randomness for non-computable measures, as well as blind weak 2-randomness.

W2R wrt a semi-measure is promising, 1

Given a left-c.e. semi-measure ρ, a *generalized ρ-test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 subsets of 2^ω such that for each i,

$$\lim_{i\to\infty} \rho(\mathcal{U}_i) = 0.$$

Theorem (BHPS)

$X \in \mathrm{bW2R}_{\overline{\rho}}$ if and only if for every generalized ρ-test $(\mathcal{U}_i)_{i\in\omega}$, $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$.

W2R wrt a semi-measure is promising, 2

Unlike $\mathrm{bMLR}_{\overline{\rho}}$, we have preservation of randomness for $\mathrm{bW2R}_{\overline{\rho}}$:

Theorem (BHPS)

If $X \in \mathrm{W2R}$ and Φ is a Turing functional such that $X \in \mathrm{dom}(\Phi)$, then $\Phi(X) \in \mathrm{bW2R}_{\overline{\rho}}$.

5. Open questions

Question

If Φ and Ψ are Turing functionals that induce the same semi-measure, i.e.,

$$\lambda_\Phi = \lambda_\Psi,$$

does it follow that $\Phi(\text{W2R}) = \Psi(\text{W2R})$?

Question

If $Y \in \text{bW2R}_{\overline{\rho}}$ and $\rho = \lambda_\Phi$ for some Turing functional Φ, is there some $X \in \text{W2R}$ such that $\Phi(X) = Y$?

Question

For a given left-c.e. semi-measure ρ, how complicated can the set of Turing degrees of representations of $\overline{\rho}$ be?