Algorithmic Randomness and Semi-Measures

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Logic Colloquium 2013
Évora, Portugal
26 July 2013

A motivating question

Suppose we have an algorithmic procedure P that, upon receiving an infinite binary sequence as input, outputs either an infinite binary sequence or a finite binary string.

$\mathcal{Q}$: What is the 'typical' infinite output of P?

Towards providing a partial answer

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Martin-Löf randomness

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 (i.e. effectively open) subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- A sequence $X \in 2^\omega$ *passes the μ-Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes *every* μ-Martin-Löf test.

Induced measures

Let us say that a Turing functional Φ is almost total if $\lambda(\mathrm{dom}(\Phi)) = 1$.

Given an almost total Turing functional Φ, the measure induced by Φ is defined by

$$\lambda_\Phi(\mathcal{S}) = \lambda(\{X \in 2^\omega : \Phi(X) \in \mathcal{S}\}).$$

for all measurable $\mathcal{S} \subseteq 2^\omega$.

Moreover, λ_Φ is a computable measure.

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- For every Martin-Löf random input X, $\Phi(X)$ is Martin-Löf random with respect to λ_Φ. (randomness preservation)
- For every sequence Y that is Martin-Löf random with respect to λ_Φ, there is some Martin-Löf random X such that $\Phi(X) = Y$. (the ex nihilo principle)

But we don't just want a partial answer!

This is the question that we want to answer:

$\mathcal{Q}^*$: If Φ yields an infinite output *with probability strictly less than one*, what is the 'typical' infinite output of Φ?

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λ_Φ is not a measure, but rather a semi-measure.

In addition, λ_Φ is left-c.e., which means that the values $\lambda_\Phi(\sigma)$ for $\sigma \in 2^{<\omega}$ are uniformly computably approximable from below.

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$\mathcal{Q}^*$: If Φ yields an infinite output with probability strictly less than one, what is the 'typical' infinite output of Φ?

To answer $\mathcal{Q}^*$ along the same lines as our partial answer to $\mathcal{Q}$ for almost total functionals, we need to answer the following:

$\mathcal{Q}^{**}$: Which sequences are random with respect to a left-c.e. semi-measure?

A naive approach

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Let's say a *ρ-test* is a uniform sequence $(U_i)_{i \in \omega}$ of c.e. subsets of $2^{<\omega}$ such that for each i,

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Can we define randomness with respect to a semi-measure in terms of ρ-tests?

The drawback of ρ-tests

Unfortunately, ρ-tests don't behave so nicely:

Proposition (BHPS)

There is a left-c.e. semi-measure ρ such that for any uniform sequence $(U_i)_{i\in\omega}$ of c.e. subsets of $2^{<\omega}$ satisfying

$$\rho(U_i) \leq 2^{-i}$$

for every $i \in \omega$, we have $\bigcap_{i\in\omega} [\![U_i]\!] = \emptyset$.

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does it follow that $\Phi(\mathrm{MLR}) = \Psi(\mathrm{MLR})$?

A positive answer to Shen's question would justify the following definition:

Y is random with respect to a semi-measure ρ if for any Turing functional Φ such that $\rho = \lambda_\Phi$, there is some $X \in \mathrm{MLR}$ such that $\Phi(X) = Y$.

A negative answer to Shen's question

But we have the following.

Proposition (BHPS)

There exist Turing functionals Φ *and* Ψ *such that*

$$\lambda_\Phi = \lambda_\Psi$$

and

$$\Phi(\mathrm{MLR}) \neq \Psi(\mathrm{MLR}).$$

Trimming a semi-measure back to a measure

If ρ is a left-c.e. semi-measure, we can define

$$\overline{\rho}(\sigma) := \inf_n \sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau).$$

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One can verify that $\overline{\rho}$ is the largest measure such that $\overline{\rho} \leq \rho$ (but it is not a probability measure in general).

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As we will see shortly, $\overline{\rho}$ can be computationally unwieldy.

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- We can allow access to $\overline{\rho}$ as an oracle in enumerating our tests.
- We can enumerate our tests without access to $\overline{\rho}$ as an oracle.

This latter approach is referred to as blind Martin-Löf randomness.

Encoding information in $\overline{\rho}$

Theorem (BHPS)

There is a left-c.e. semi-measure ρ and some $\alpha \in (0,1)$ such that

- $\overline{\rho} = \alpha \cdot \lambda$; *and*
- $\alpha \equiv_T \emptyset''$.

Some consequences

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Blind Martin-Löf randomness with respect to $\overline{\rho}$ seems promising.

Problem: The preservation of blind randomness is not satisfied for some $\overline{\rho}$ derived from a left-c.e. semi-measure ρ.

Weak 2-randomness

Definition

Let μ be a computable measure.

- A *generalized μ-Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 (i.e. effectively open) subsets of 2^ω such that

$$\lim_{i\to\infty} \mu(\mathcal{U}_i) = 0.$$

- $X \in 2^\omega$ is *μ-weakly 2-random*, denoted $X \in \mathrm{W2R}_\mu$, if X passes *every* μ-Martin-Löf test.

We can also define weak 2-randomness for non-computable measures such as $\overline{\rho}$, as well as blind weak 2-randomness.

The virtues of W2R wrt a semi-measure, 1

Given a left-c.e. semi-measure ρ, a *generalized ρ-test* is a uniform sequence $(U_i)_{i\in\omega}$ of c.e. subsets of $2^{<\omega}$ such that for each i,

$$\lim_{i\to\infty} \rho(U_i) = 0.$$

Theorem (BHPS)

Let ρ be a left-c.e. semi-measure. Then X passes every generalized ρ-test if and only X is blind weakly 2-random with respect to $\overline{\rho}$.

The virtues of W2R wrt a semi-measure, 2

Unlike blind Martin-Löf randomness with respect to $\overline{\rho}$, we have preservation of randomness for blind weak 2-randomness with respect to $\overline{\rho}$.

Theorem (BHPS)

If $X \in \mathrm{W2R}$ and Φ is a Turing functional such that $X \in \mathrm{dom}(\Phi)$, then $\Phi(X)$ is blind weakly 2-random with respect to $\overline{\rho}$.

The virtues of W2R wrt a semi-measure, 2

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Theorem (BHPS)

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Open question: Does blind weak 2-randomness with respect to $\overline{\rho}$ satisfy the ex nihilo principle?