Initial segment complexity and randomness for computable measures

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Introduction

According to the Levin-Schnorr theorem, a sequence $X \in 2^\omega$ is Martin-Löf random with respect to the Lebesgue measure if and only if X has sufficiently high initial segment complexity.

Although a similar result also holds for sequences that are random with respect to some computable measure (I refer to such sequences as proper sequences), the growth rates of the initial segment complexity of proper sequences can vary quite widely.

Introduction, continued

The goal of the talk today is to discuss

- the various growth rates of the initial segment complexity of proper sequences; and
- the extent to which properties of a computable measure μ are reflected in the initial segment complexity of sequences random with respect to μ.

Outline of the talk

1. Background
2. Random sequences of high initial segment complexity
3. Random sequences of low initial segment complexity

1. Background

Computable measures on 2^ω

For $\sigma \in 2^{<\omega}$, let $[\![\sigma]\!] = \{X \in 2^\omega : \sigma \prec X\}$.

Definition

A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu([\![\sigma]\!])$ is computable as a real-valued function.

In other words, μ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that

$$|\mu([\![\sigma]\!]) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

From now on we will write $\mu(\sigma)$ instead of $\mu([\![\sigma]\!])$.

Martin-Löf randomness with respect to a computable measure

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ *passes* a μ-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$.
- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes every μ-Martin-Löf test.

We will say that X is *proper* if $X \in \mathrm{MLR}_\mu$ for some computable measure μ on 2^ω.

Kolmogorov complexity

Let $U : 2^{<\omega} \to 2^{<\omega}$ be a universal, prefix-free Turing machine.

For each $\sigma \in 2^{<\omega}$, the *prefix-free Kolmogorov complexity* of σ is defined to be

$$K(\sigma) := \min\{|\tau| : U(\tau)\downarrow = \sigma\}.$$

The Levin-Schnorr Theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ is Martin-Löf random if and only if

$$\forall n\, K(X \upharpoonright n) \geq n - O(1).$$

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More generally, we have the following:

The Levin-Schnorr Theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ is Martin-Löf random if and only if

$$\forall n\, K(X\upharpoonright n) \geq n - O(1).$$

More generally, we have the following:

Theorem

Let μ be a computable measure. $X \in 2^\omega$ is μ-Martin-Löf random if and only if

$$\forall n\, K(X\upharpoonright n) \geq -\log(\mu(X\upharpoonright n)) - O(1).$$

Atomic Measures and Continuous Measures

A measure μ on 2^ω is *atomic* if there is some $A \in 2^\omega$ such that $\mu(\{A\}) > 0$.

A is called an *atom* of μ.

For an atomic measure μ, let Atoms_μ be the collection of atoms of μ.

If μ is not atomic, then μ is *continuous*.

A few facts:

- If A is the atom of a computable measure, then $A \in \mathrm{MLR}_\mu$.
- If A is the atom of a computable measure, then A is computable.

A priori complexity

Definition

- A *semi-measure* is a function $\rho : 2^{<\omega} \to [0,1]$ satisfying
  (i) $\rho(\epsilon) = 1$ and
  (ii) $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.
- A semi-measure ρ is *left-c.e.* if ρ is computably approximable from below.

Fact: There exists a *universal* left-c.e. semi-measure M. That is, for every left-c.e. semi-measure ρ there is some c such that

$$c \cdot M(\sigma) \geq \rho(\sigma)$$

for every σ.

We define the *a priori complexity* of $\sigma \in 2^{<\omega}$ to be

$$KA(\sigma) := -\log M(\sigma).$$

Complex and strongly complex sequences

Recall that an order function $h : \omega \to \omega$ is an unbounded, non-decreasing function.

Definition

Let $X \in 2^\omega$.

- X is *complex* if there is a computable order function $h : \omega \to \omega$ such that

$$\forall n\, K(X{\upharpoonright}n) \geq h(n).$$

- X is *strongly complex* if there is a computable order function $g : \omega \to \omega$ such that

$$\forall n\, KA(X{\upharpoonright}n) \geq g(n).$$

Proposition

X is complex if and only if X is strongly complex.

2. Random sequences with high initial segment complexity

What counts as high initial segment complexity?

In what follows, we will consider a proper sequence to have high initial segment complexity if it is complex.

It is worth noting that not every complex sequence is proper.

For example, there is a complex sequence of minimal Turing degree, but no proper sequence has minimal Turing degree.

A preliminary observation

Suppose that X is Martin-Löf random with respect to a computable measure μ.

Then by the Levin-Schnorr theorem,

$$\forall n\ K(X\upharpoonright n) \geq -\log(\mu(X\upharpoonright n)) - O(1).$$

Note that this does not imply that X is complex, since the function $n \mapsto -\log(\mu(X\upharpoonright n))$ is in most cases not computable but only X-computable.

A sufficient condition for complexity

Theorem (Hölzl, Merkle, Porter)

If $X \in 2^\omega$ is Martin-Löf random with respect to a computable, continuous measure μ, then X is complex.

A sufficient condition for complexity

Theorem (Hölzl, Merkle, Porter)

If $X \in 2^\omega$ is Martin-Löf random with respect to a computable, continuous measure μ, then X is complex.

This follows from the following two results.

Lemma

Let μ be a computable, continuous measure and let $X \in \mathsf{MLR}_\mu$. Then there is some Martin-Löf random $Y \leq_{\mathrm{tt}} X$.

Lemma

If Y is complex and $Y \leq_{\mathrm{wtt}} X$, then X is complex.

What about the converse?

The converse of the previous theorem doesn't hold: as stated earlier, there are complex sequences that are not proper.

However, we do have a partial converse.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

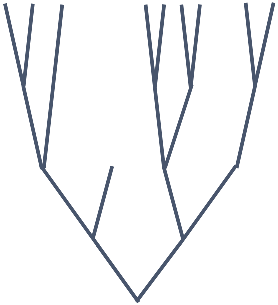
A useful lemma

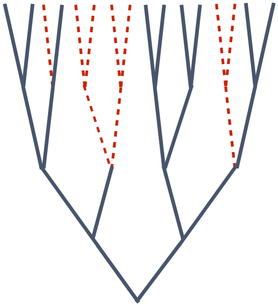
Lemma

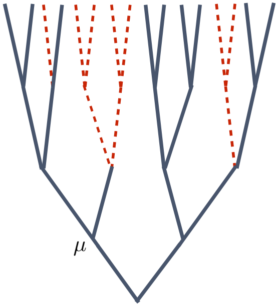
Suppose that

- μ *is a computable measure,*
- $X \in \mathsf{MLR}_\mu$ *is non-computable,*
- $\mathcal{P}$ *is a* Π^0_1 *class with no computable members, and*
- $X \in \mathcal{P}$.

Then there is some computable, continuous measure ν *such that* $X \in \mathsf{MLR}_\nu$.



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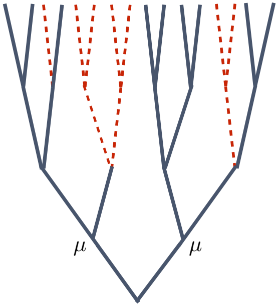
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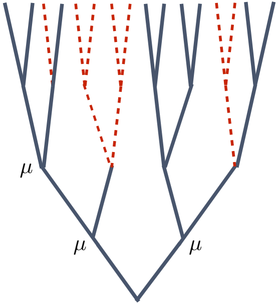
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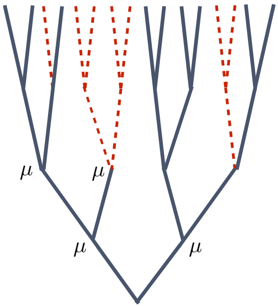
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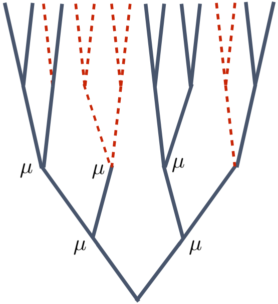

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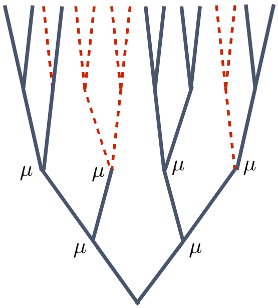

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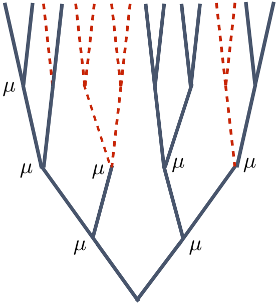
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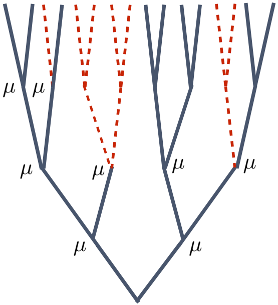
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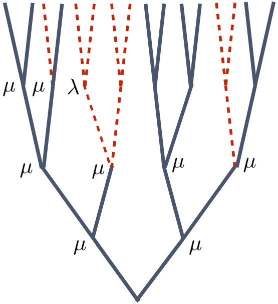
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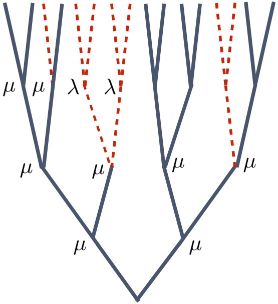
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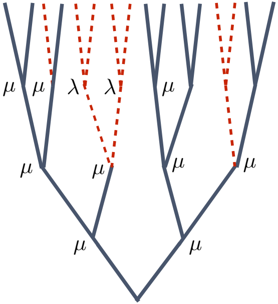
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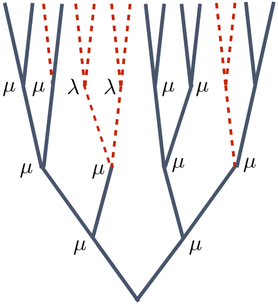
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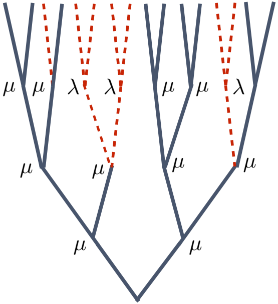
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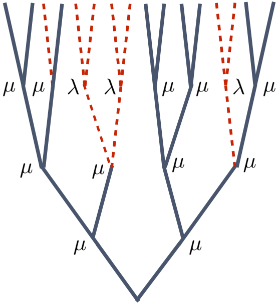

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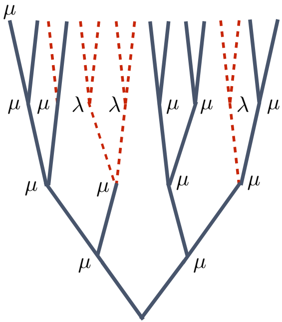

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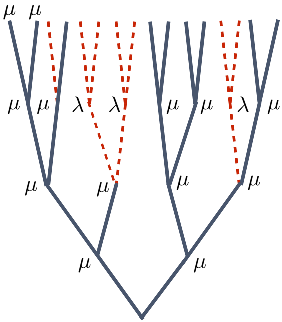

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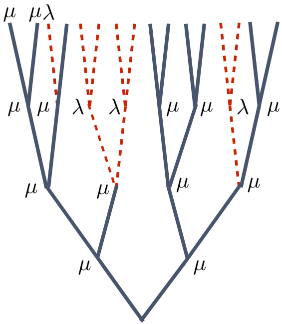
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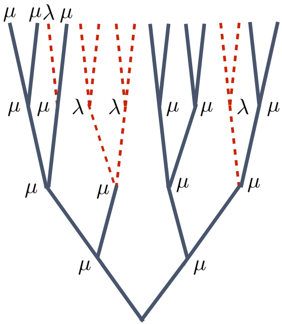

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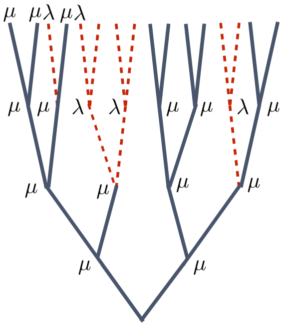
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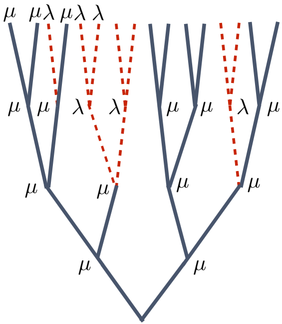
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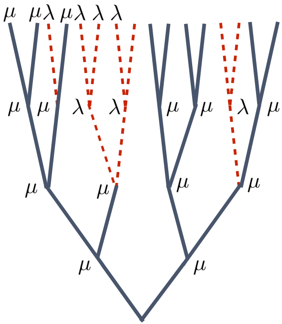
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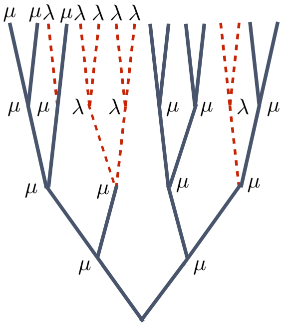

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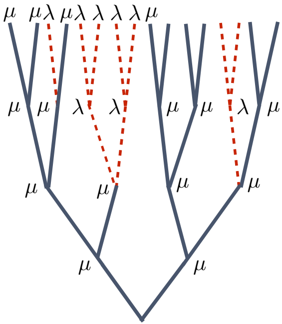
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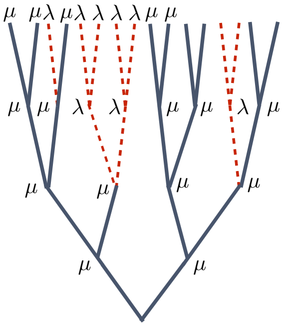

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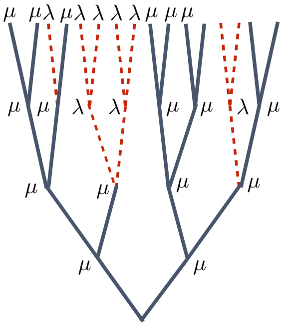
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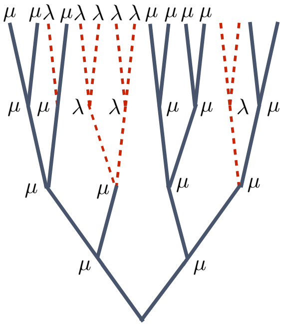

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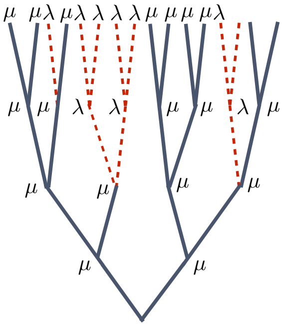
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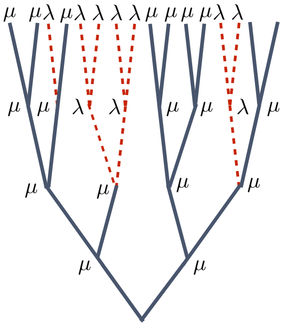
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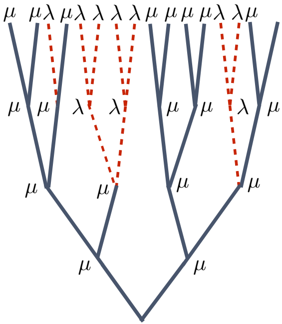
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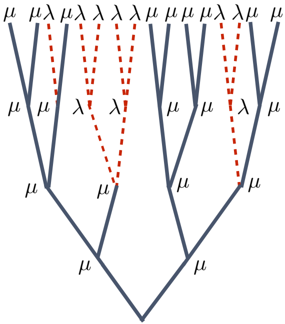

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Establishing the partial converse

Theorem

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

Establishing the partial converse

Theorem

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

To prove this theorem, let h be the computable order function that witnesses that X is complex.

Then we apply the previous lemma to the Π^0_1 class

$$\{A \in 2^\omega : K(A{\restriction}n) \geq h(n)\},$$

which contains X but no computable sequences.

Connection to semigenericity

Definition

$X \in 2^\omega$ is *semigeneric* if for every Π^0_1 class $\mathcal{P}$ with $X \in \mathcal{P}$, $\mathcal{P}$ contains some computable member.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. $X \in \mathrm{MLR}_\mu$ *for some computable, continuous* μ.
2. X *is complex.*
3. X *is not semigeneric.*

Avoidability and hyperavoidability

Definition

(i) $X \in 2^\omega$ is *avoidable* if there is some partial computable function p, called an *avoidance function*, such that for every computable set M and every index e for M, $p(e)\downarrow$ and $X \upharpoonright p(e) \neq M \upharpoonright p(e)$.

(ii) Moreover, X is *hyperavoidable* if X is avoidable with a total avoidance function.

- Not every avoidable sequence is hyperavoidable.
- X is hyperavoidable if and only if X is complex.
- A non-computable sequence X is avoidable if and only if X is not semigeneric.

Additional consequences

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. $X \in \mathrm{MLR}_\mu$ *for some computable, continuous* μ.
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Additional consequences

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. $X \in \mathrm{MLR}_\mu$ *for some computable, continuous* μ.
2. X *is complex.*
3. X *is not semigeneric.*
4. X *is hyperavoidable.*
5. X *is avoidable.*

A follow-up question

Let μ be a computable, continuous measure.

Since every sequence that is random with respect μ is complex, is there a single computable order function that witnesses the complexity of μ-random sequences?

Is there a least such function (up to an additive constant)?

A follow-up result

Definition

Let μ be a continuous measure. Then the *granularity function of μ*, denoted g_μ, is the order function mapping n to the least ℓ such that $\mu(\sigma) < 2^{-n}$ for every σ of length ℓ.

A follow-up result

Definition

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Theorem (Hölzl, Merkle, Porter)

Let μ be a computable, continuous measure and let $X \in \mathrm{MLR}_\mu$. Then we have

$$\forall n \; KA(X{\upharpoonright}n) \geq g_\mu^{-1}(n) - O(1).$$

Some facts about the granularity of a computable measure

- If μ is exactly computable, that is, μ is $\mathbb{Q}_2$-valued and the function $\sigma \mapsto \mu(\sigma)$ is a computable function, then g_μ is computable.

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Some facts about the granularity of a computable measure

- If μ is exactly computable, that is, μ is $\mathbb{Q}_2$-valued and the function $\sigma \mapsto \mu(\sigma)$ is a computable function, then g_μ is computable.
- However, there is a computable, continuous measure μ such that the granularity function g_μ of μ is not computable.
- For every computable, continuous measure μ, there is a computable order function $f : \omega \to \omega$ such that

$$|f(n) - g_\mu(n)^{-1}| \leq O(1).$$

Such a function f provides as a global computable lower bound for the initial segment complexity of every μ-random sequence.

A question about uniformity

Question

If we have a computable, atomic measure μ such that

$$\forall X \in 2^\omega \ (X \in \mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \ \Rightarrow \ X \text{ is complex}),$$

is there a computable, continuous measure ν such that

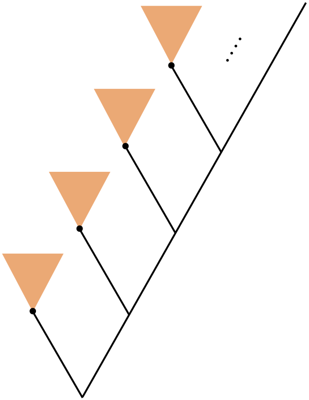
$$\mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \subseteq \mathsf{MLR}_\nu?$$

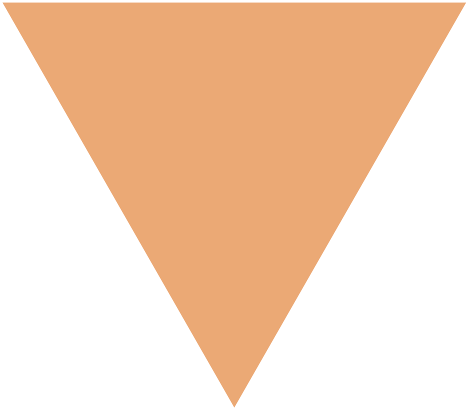
An answer

Theorem (Hölzl, Merkle, Porter)

There is a computable, atomic measure μ such that

- *every* $X \in \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu$ *is complex but*
- *there is no computable, continuous measure* ν *such that* $\mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \subseteq \mathrm{MLR}_\nu$.





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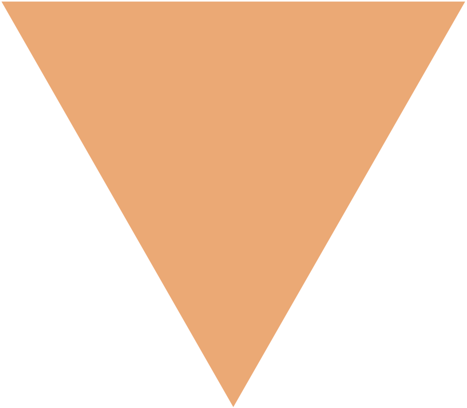
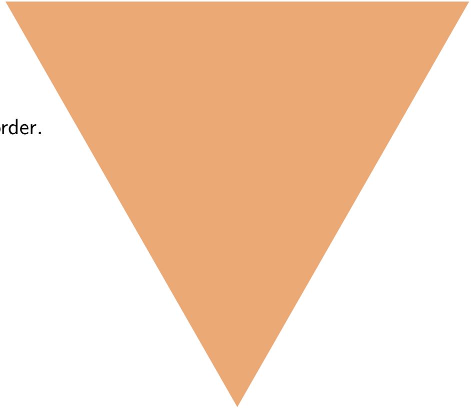


the i^{th} neighborhood

Suppose that ϕ_i is an order.



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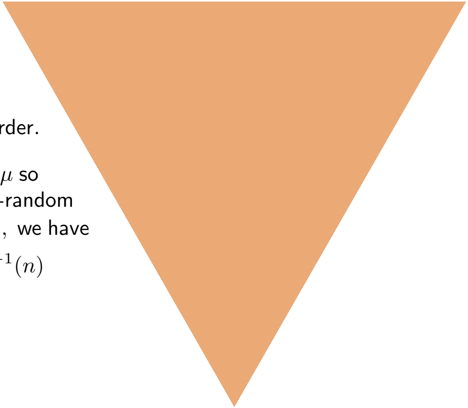

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We define the measure μ so that for any complex μ-random X in this neighborhood, we have

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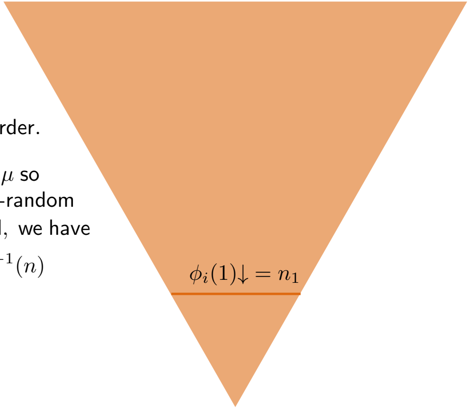

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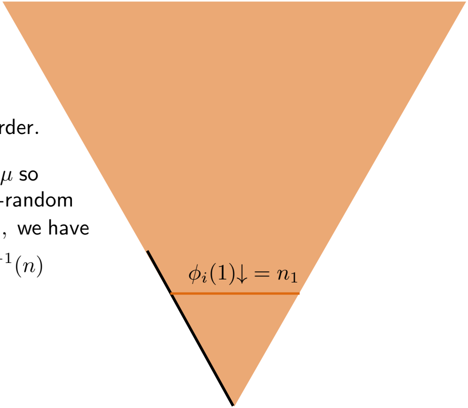

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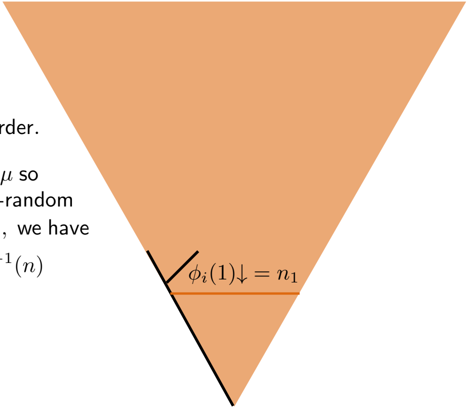

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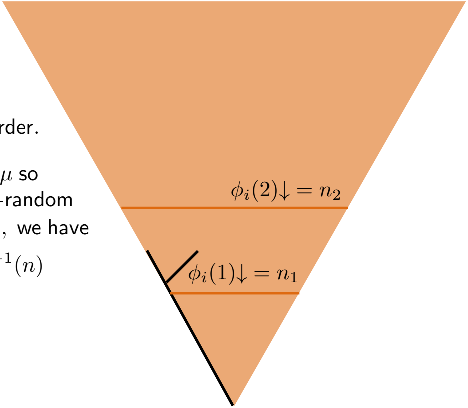

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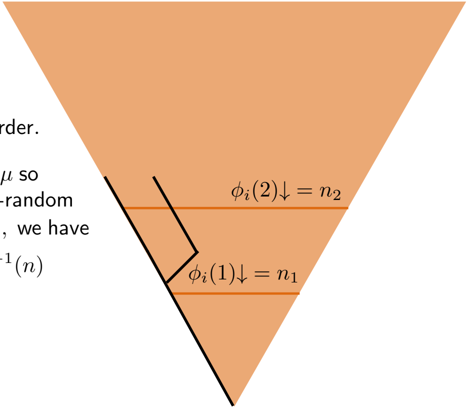

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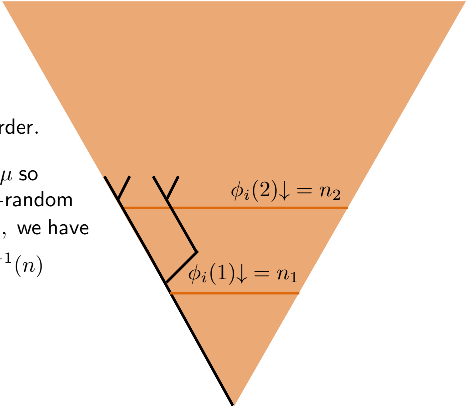

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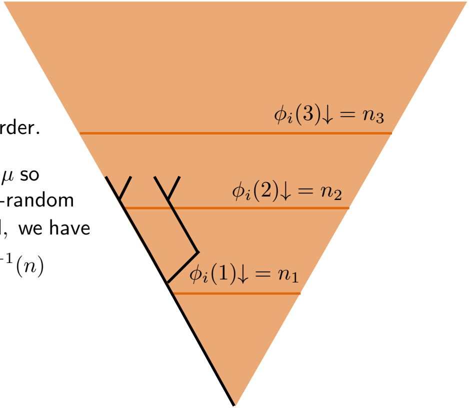

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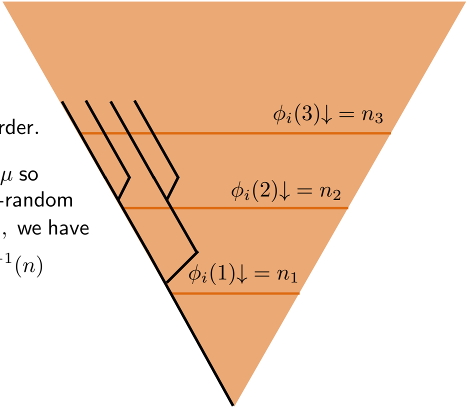

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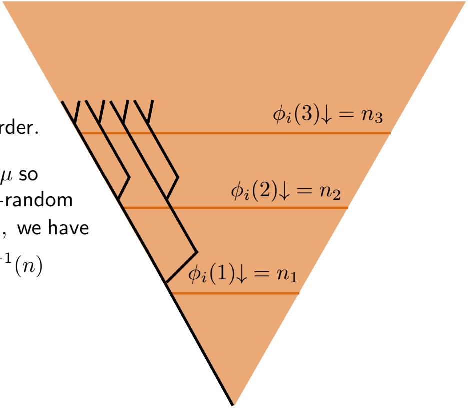

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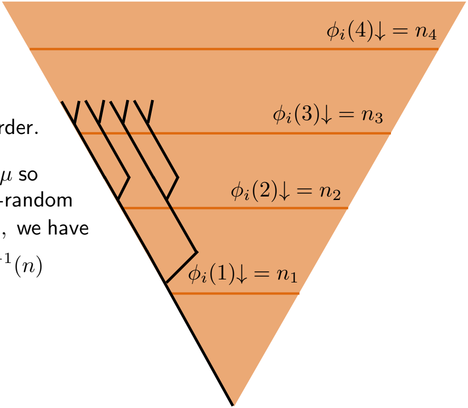

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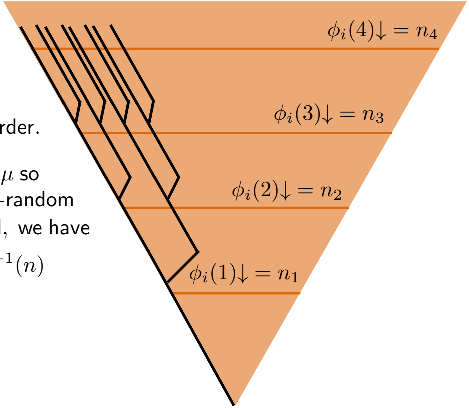

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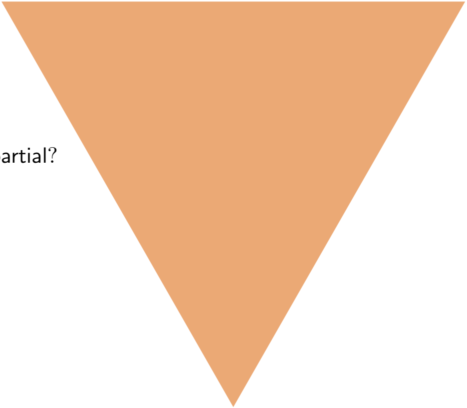

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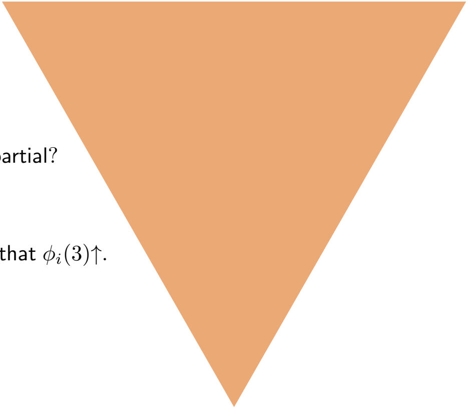

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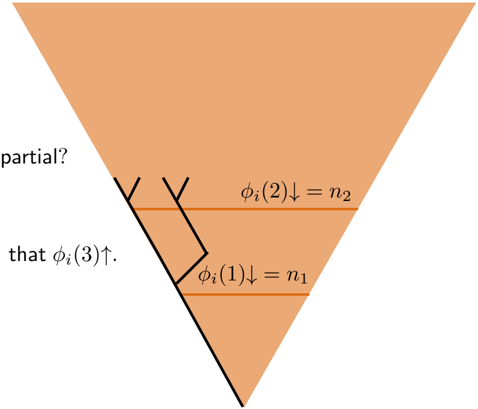

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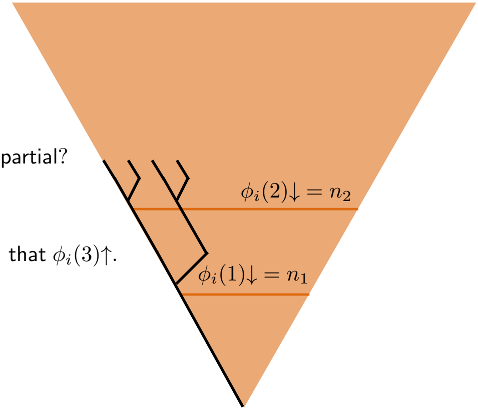

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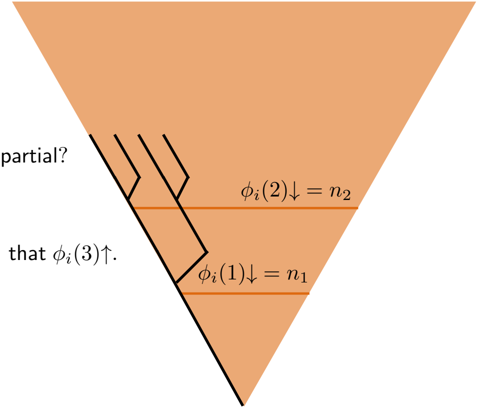

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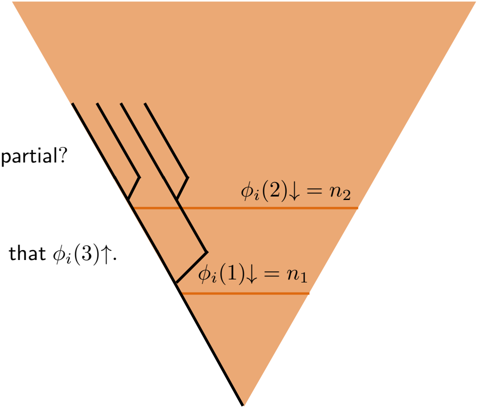

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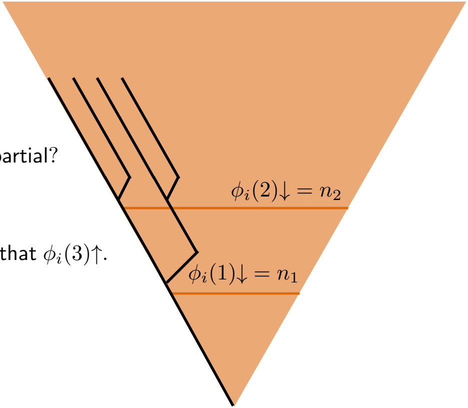

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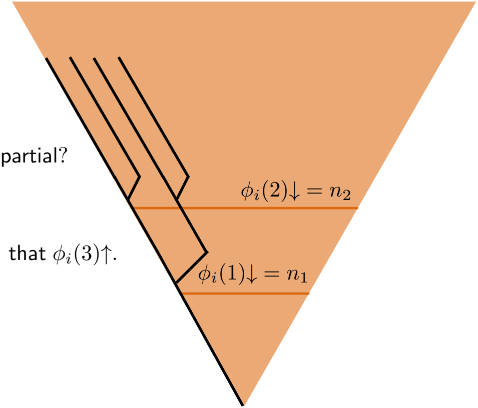

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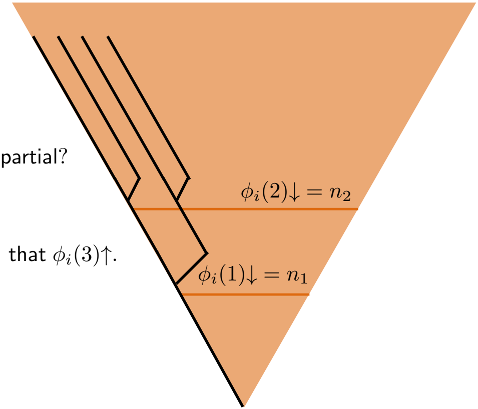

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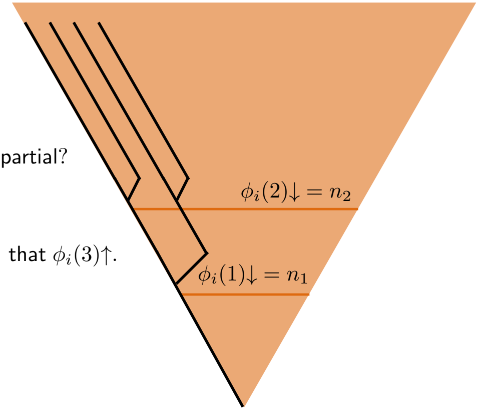

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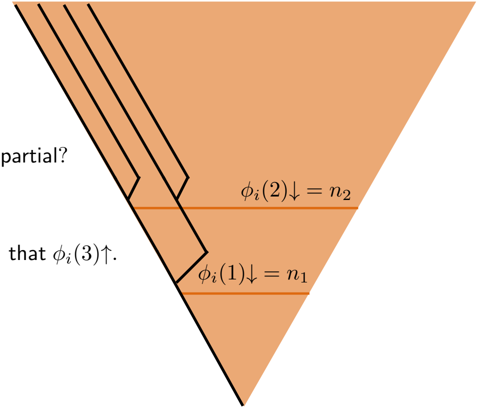

Let $[\![\sigma_i]\!]$ be the i^{th} neighborhood.

One can verify that

- if ϕ_i is partial, then $[\![\sigma_i]\!] \cap \mathrm{MLR}_\mu \subseteq \mathrm{Atoms}_\mu$;
- if ϕ_i is total, then $[\![\sigma_i]\!] \cap \mathrm{Atoms}_\mu = \emptyset$ and every $X \in \mathrm{MLR}_\mu \cap [\![\sigma_i]\!]$ is complex.

Lastly, if there is some computable, continuous ν such that $\mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \subseteq \mathrm{MLR}_\nu$, then there is a computable order $f = \phi_i$ such that for every $X \in \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu$,

$$KA(X{\upharpoonright}n) \geq f^{-1}(n) - O(1)$$

for every n, which yields a contradiction.

3. Random sequences with low initial segment complexity

Notions of non-complexity

Definition

(i) X is *infinitely often complex* (or *i.o. complex*) if there is some computable order function f such that $K(X \upharpoonright f(n)) \geq n$ for infinitely many n.

(ii) X is *anti-complex* if for every computable order function f we have $K(X \upharpoonright f(n)) \leq n$ for almost every n.

(iii) X is *infinitely often anti-complex* (or *i.o. anti-complex* if for every computable order function f we have $K(X \upharpoonright f(n)) \leq n$ for infinitely every n.

not complex $\Rightarrow$ i.o. anti-complex
not anti-complex $\Rightarrow$ i.o. complex

KA-versions of non-complexity

Each of the notions on the previous slide can equivalently be formulated in terms of a priori complexity (KA).

One benefit of working with KA rather than K in this context is given by the following result, which does not hold for K.

Lemma

$X \in 2^\omega$ is anti-complex if and only if for every computable order f, $KA(X\restriction n) \leq f(n) + O(1)$.

Proper non-complex sequences?

By our earlier result, if a proper sequence is not random with respect to any continuous, computable measure, it cannot be complex and must be i.o. anti-complex.

Do such proper sequences exist?

That is, are there proper sequences that are only random with respect to atomic computable measures?

Tally functionals

The way that we construct proper sequences that are not complex is to use *tally functionals*.

Roughly, a tally functional Φ is a total Turing functional that maps each $X \in 2^\omega$ to a sequence of the form

$$1^{f(0)}\, 0\, 1^{f(1)}\, 0\, 1^{f(2)} \dots$$

where f is usually some sufficiently fast growing function.

Moreover, for some sequences X we may have $\Phi(X) = \sigma 1^\omega$.

Tally functionals (continued)

Given a Martin-Löf random $X \in 2^\omega$, $\Phi(X)$ will be random with respect to the measure induced by Φ, λ_Φ, defined to be

$$\lambda_\Phi(\mathcal{X}) = \lambda(\Phi^{-1}(\mathcal{X})).$$

Moreover, for specific choices of tally functional Φ and sequence X, the sequence $\Phi(X)$ will be non-computable and not complex.

I.o. anti-complex proper sequences

In earlier work with Bienvenu, we showed:

Theorem (Bienvenu, Porter)

Let **a** *be a random Turing degree. Then* **a** *contains an i.o. anti-complex proper sequence if and only if* **a** *is hyperimmune.*

With some additional work, this can be slightly improved.

Theorem (Hölzl, Merkle, Porter)

Let **a** *be a random Turing degree. Then* **a** *contains an i.o. anti-complex, i.o. complex proper sequence if and only if* **a** *is hyperimmune.*

Anti-complex proper sequences

A similar result holds for anti-complex proper sequences.

Theorem (Hölzl, Merkle, Porter)

Let $\mathbf{a}$ *be a random Turing degree. Then* $\mathbf{a}$ *contains an anti-complex proper sequence if and only if* $\mathbf{a}$ *is high.*

The $(\Leftarrow)$ direction is shown by a tally functional construction.

The ($\Rightarrow$) direction

For the other direction, suppose that X is an anti-complex proper sequence.

Let f be a computable order function, so that f^{-1} is also a computable order function. By our earlier lemma,

$$KA(X\upharpoonright n) \leq f^{-1}(n) + O(1).$$

In addition, since X is proper, there is some computable measure μ such that

$$KA(X\upharpoonright n) \geq -\log(\mu(X\upharpoonright n)) + O(1).$$

Combining these two inequalities, yields

$$-\log(\mu(X\upharpoonright n)) \leq f^{-1}(n) - O(1).$$

If we set $g(n) = -\log(\mu(X\upharpoonright n))$, one can show that the function $g^{-1}(2n)$ dominates f.

Thank you!