Algorithmic randomness for non-uniform probability measures

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Introduction

In algorithmic randomness, a sub-discipline of computability theory, one major research focus is to study the relationships between various formal definitions of randomness.

In this talk, I will focus primarily on two equivalent definitions of random infinite sequence:

- Kolmogorov incompressible sequences, and
- Martin-Löf random sequences.

The equivalence of these two definitions, known as the Levin-Schnorr theorem, is one of the central results in the theory of algorithmic randomness.

Introduction (continued)

The goals of today's talk are to:

- motivate and precisely define these two notions of randomness;
- outline the proof of their equivalence;
- extend these definitions to computable probability measures on 2^ω; and
- to discuss some recent work on the interplay between
  (i) the growth rates of the initial segment complexity of sequences random with respect to some computable probability measure, and
  (ii) certain properties of this underlying measure (such as continuity vs. discontinuity).

Outline

1. Definitions of algorithmic randomness
2. The Levin-Schnorr theorem
3. Randomness with respect to a computable measure
4. The initial segment complexity of proper sequences

1. Definitions of algorithmic randomness

A motivating question

What does it mean for a sequence of 0s and 1s to be random?

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(3) List names of American states alphabetically: $0 =$ even $\#$ of letters, $1 =$ odd $\#$ of letters.

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(3) List names of American states alphabetically: $0 =$ even # of letters, $1 =$ odd # of letters.

(4) First fifty digits of the binary expansion of π.

(5) Fifty digits obtained from random.org (atmospheric noise?).

Two rough definitions of algorithmic randomness

Intuitively, a sequence is algorithmically random if it contains no "effectively definable regularities."

"effectively definable regularities" $\approx$ patterns definable in some computable way

Suppose $X \in 2^\omega$ contains no such regularities. Then:

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Suppose $X \in 2^\omega$ contains no such regularities. Then:

1. Initial segments of X cannot be compressed by an effective procedure.

2. X cannot be detected as non-random by any effective test for randomness.

Kolmogorov Complexity (relative to a prefix-free machine M)

Let $M : 2^{<\omega} \to 2^{<\omega}$ be a Turing machine that is *prefix-free*, which means that if $M(\sigma)\downarrow$ and $\sigma \prec \tau$, then $M(\tau)\uparrow$.

Definition

The *prefix-free Kolmogorov complexity of $\sigma \in 2^{<\omega}$ relative to M* is

$$K_M(\sigma) = \min\{|\tau| : M(\tau)\downarrow = \sigma\}.$$

(We set $K_M(\sigma) = \infty$ if σ is not in the range of M.)

Some remarks

Given a prefix-free machine M such that $M(\tau) = \sigma$, τ is called an *M-description* of σ.

$K_M(\sigma)$ is thus the length of the shortest M-description of σ.

We might say that σ is random relative to M if $K_M(\sigma) \approx |\sigma|$, but we want a definition of randomness that is not dependent upon our choice of M.

Question: In terms of which machine should we define randomness?

Answer: We restrict to a *universal*, prefix-free Turing machine.

Universal Prefix-Free Turing Machines

We can effectively enumerate the collection of all prefix-free Turing machines $\{M_i\}_{i \in \omega}$.

Then the function U defined by

$$U(1^e 0\sigma) \simeq M_e(\sigma)$$

for every $e \in \omega$ and every $\sigma \in 2^{<\omega}$ is a *universal prefix-free Turing machine*.

Kolmogorov complexity

Let $U : 2^{<\omega} \to 2^{<\omega}$ be a universal, prefix-free Turing machine.

For each $\sigma \in 2^{<\omega}$, the *prefix-free Kolmogorov complexity* of σ is defined to be

$$K(\sigma) := \min\{|\tau| : U(\tau)\downarrow = \sigma\}.$$

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Question: Has our worry about the choice of Turing machine been addressed?

Optimality and Invariance

Theorem (The Optimality Theorem)

Let U be a universal prefix-free Turing machine. Then for every prefix-free Turing machine M, there is some $c \in \omega$ such that

$$K_U(\sigma) \leq K_M(\sigma) + c$$

for every $\sigma \in 2^{<\omega}$.

Optimality and Invariance

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for every $\sigma \in 2^{<\omega}$.

Consequently, we have:

Theorem (The Invariance Theorem)

For every two universal Turing machines U_1 and U_2, there is some $c_{U_1,U_2} \in \omega$ such that for every $\sigma \in 2^{<\omega}$,

$$|K_{U_1}(\sigma) - K_{U_2}(\sigma)| \leq c_{U_1,U_2}.$$

Incompressible Strings

Let $c \in \omega$. If σ satisfies

$$K(\sigma) \geq |\sigma| - c,$$

then we say that σ is *c-incompressible*.

Can this be extended to infinite sequences?

Incompressible Strings

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Can this be extended to infinite sequences?

Definition

We say that $X \in 2^\omega$ is *Kolmogorov incompressible* if

$$(\exists c)(\forall n)\ K(X \upharpoonright n) \geq n - c.$$

Lebesgue measure one many sequences are Kolmogorov incompressible.

The statistical definition of randomness (for $2^{<\omega}$)

Given a finite string $\sigma \in 2^{<\omega}$, we'd like to test whether it is random.

Null hypothesis: σ is random.

How do we test this hypothesis?

We employ a statistical test $\mathcal{T}$ that has a critical region U corresponding to the significance level α.

If our string is contained in the critical region U, we reject the hypothesis of randomness at level α (say, $\alpha = 0.05$ or $\alpha = 0.01$).

The statistical definition of randomness (for 2^ω)

Given an infinite sequence $X \in 2^\omega$, we'd like to test whether it is random.

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Given an infinite sequence $X \in 2^\omega$, we'd like to test whether it is random.

Null hypothesis: X is random.

How do we test this hypothesis?

We test initial segments of X at *every level of signficance*: $\alpha = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$

A test for 2^ω is now given by an infinite collection $(\mathcal{T}_i)_{i \in \omega}$ of tests for $2^{<\omega}$, where the critical region U_i of $\mathcal{T}_i$ corresponds to the significance level $\alpha = 2^{-i}$.

Formally. . .

A *Martin-Löf test* is a sequence $(U_i)_{i \in \omega}$ of uniformly computably enumerable sets of strings such that for each i,

$$\sum_{\sigma \in U_i} 2^{-|\sigma|} \leq 2^{-i}.$$

(Think of each U_i as the critical region for a statistical test $\mathcal{T}_i$ at significance level $\alpha = 2^{-i}$.)

A sequence $X \in 2^\omega$ *passes a Martin-Löf test* $(U_i)_{i \in \omega}$ if there is some i such that for every k, $X \upharpoonright k \notin U_i$.

$X \in 2^\omega$ is *Martin-Löf random*, denoted $X \in \mathrm{MLR}$, if X passes *every* Martin-Löf test.

The measure-theoretic formulation

Given $\sigma \in 2^{<\omega}$,

$$[\![\sigma]\!] := \{X \in 2^\omega : \sigma \prec X\}.$$

These are the basic open subsets of 2^ω.

The Lebesgue measure on 2^ω is defined by

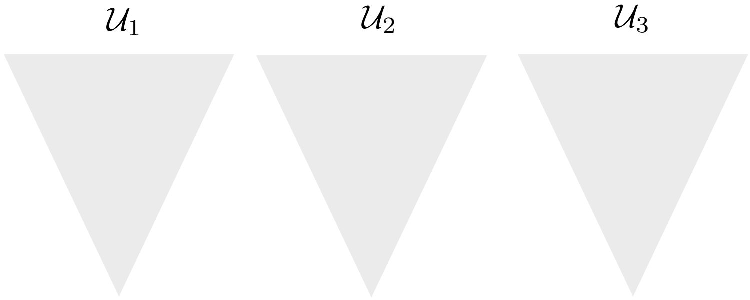
$$\lambda([\![\sigma]\!]) = 2^{-|\sigma|}.$$

Thus we can consider a Martin-Löf test to be a collection $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that

$$\lambda(\mathcal{U}_i) \leq 2^{-i}$$

for every i.

Moreover, X passes the test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.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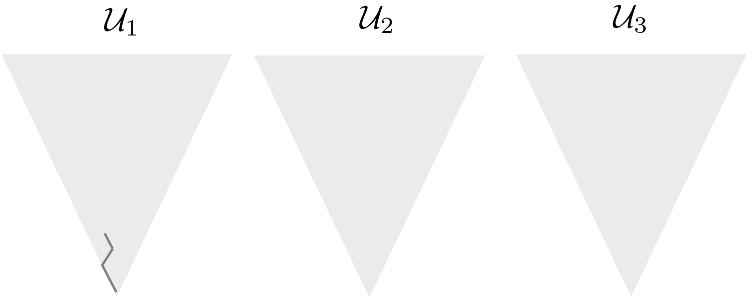
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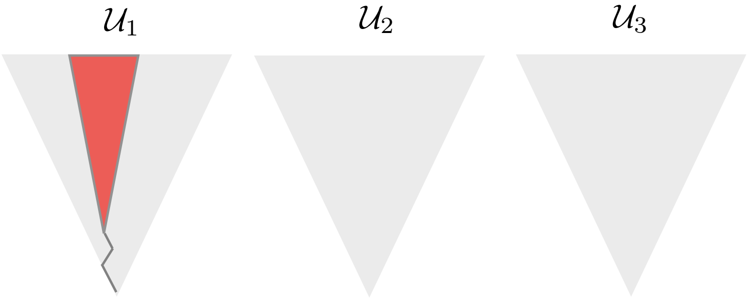
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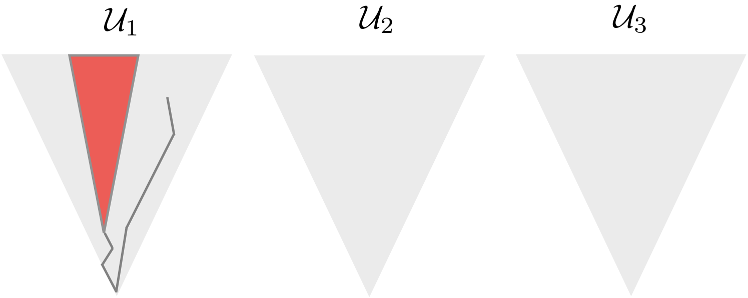
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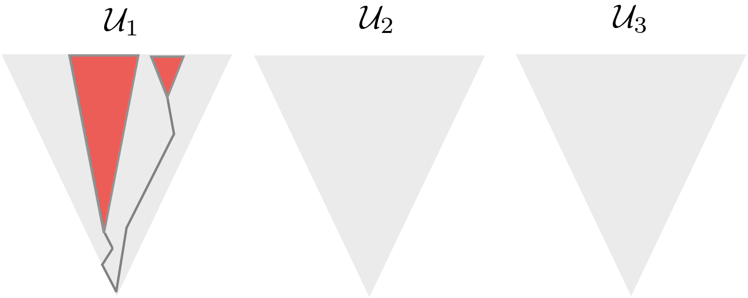
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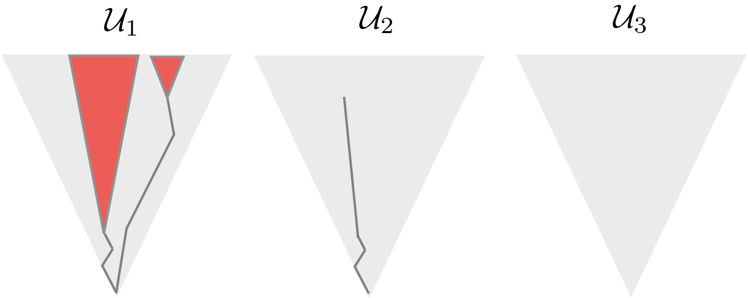
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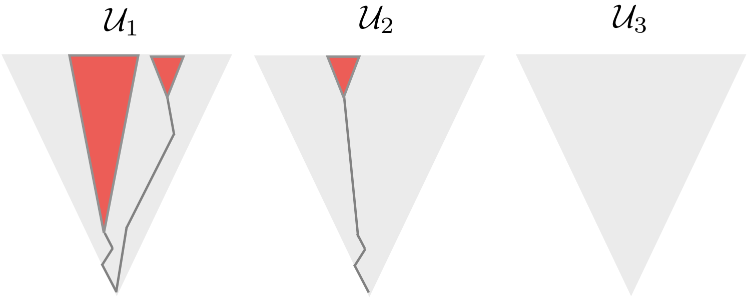
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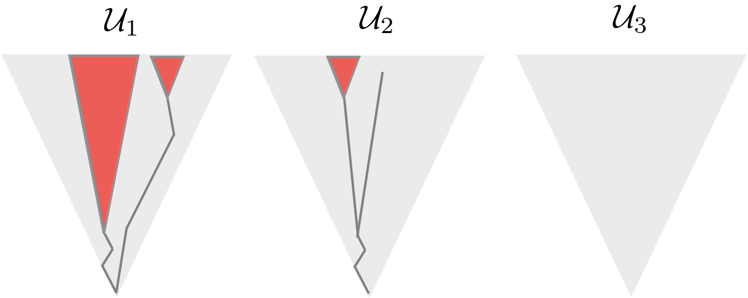
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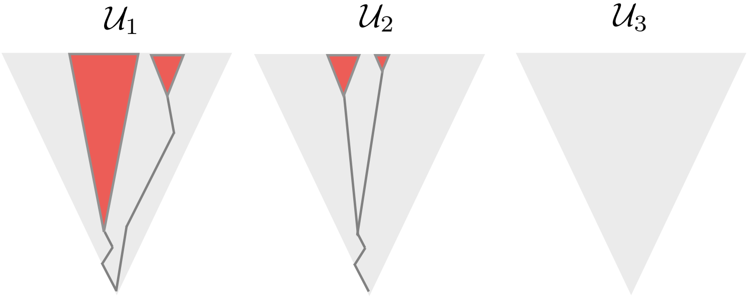
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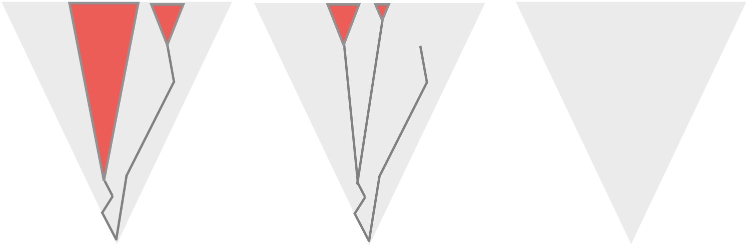

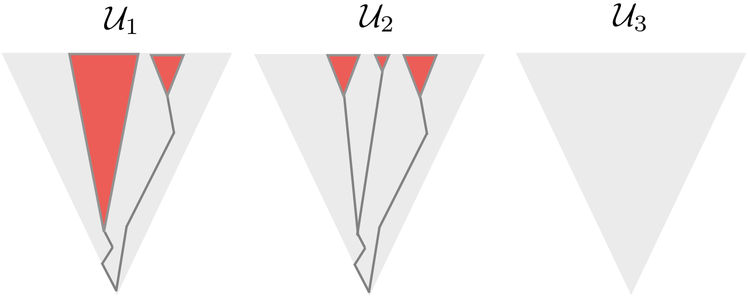
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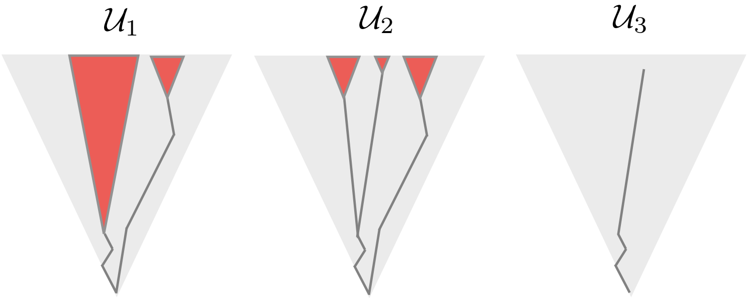

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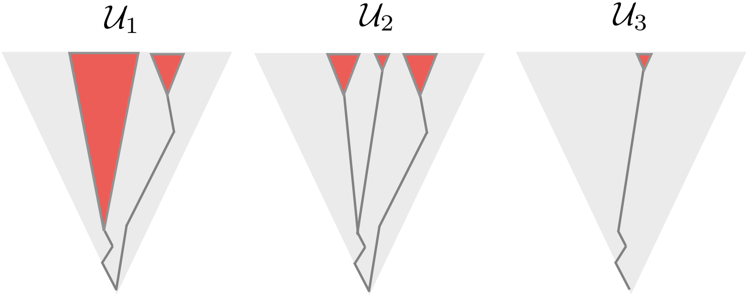
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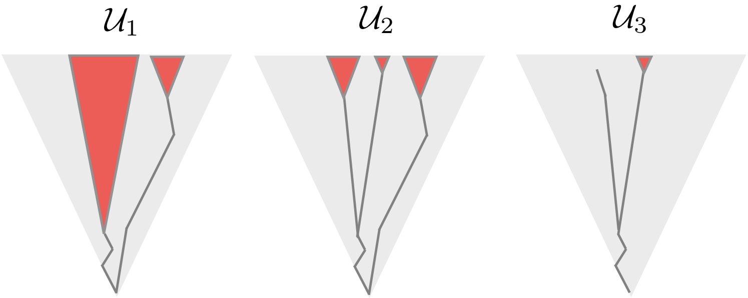
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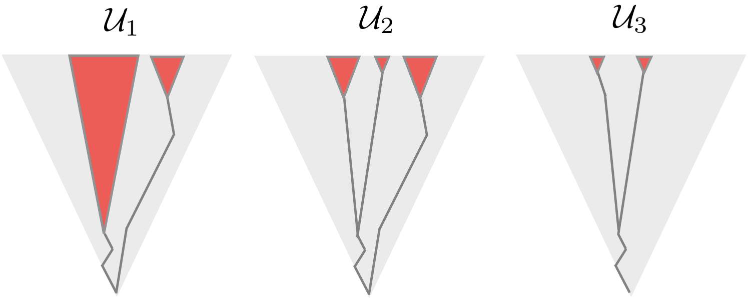
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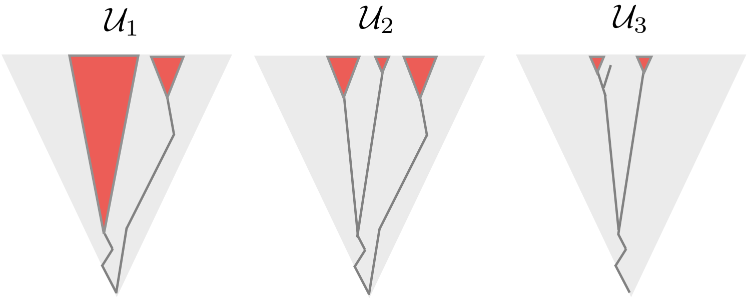
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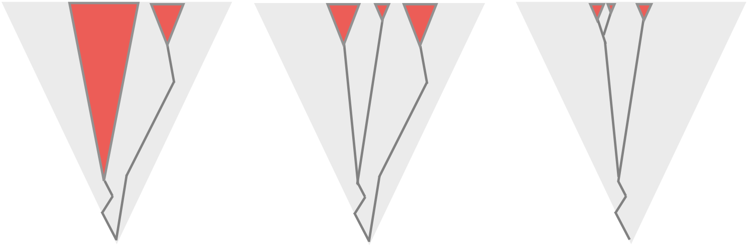

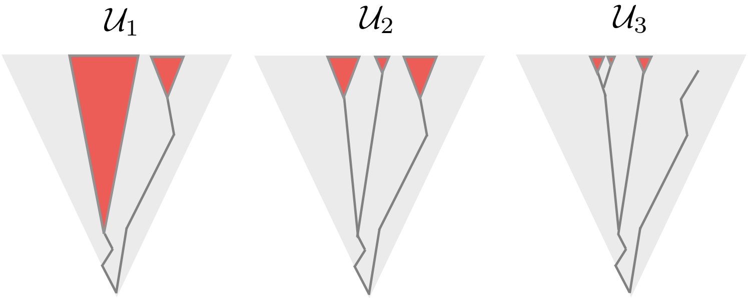
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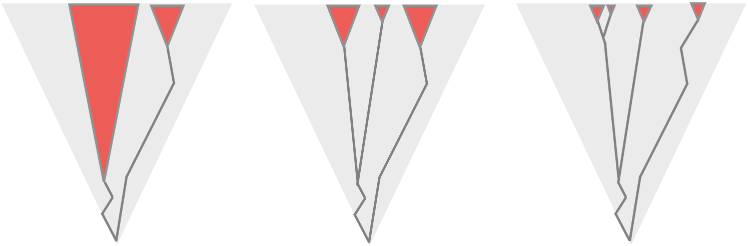

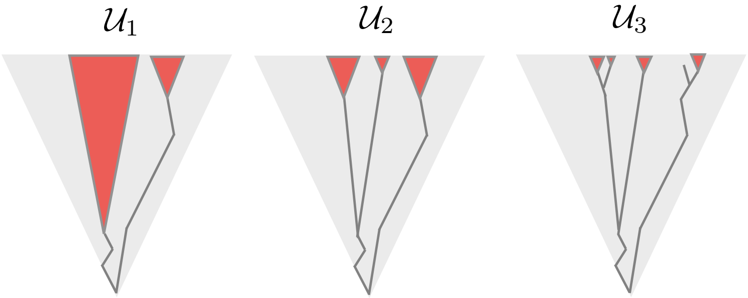
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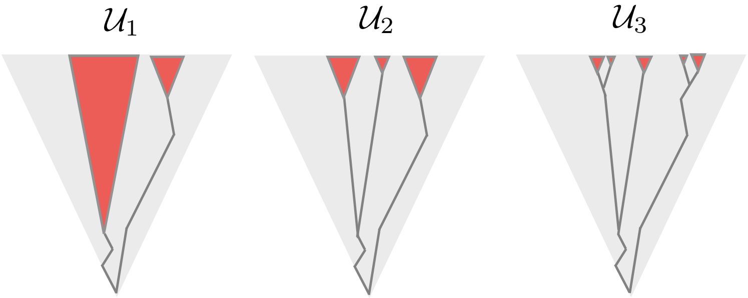
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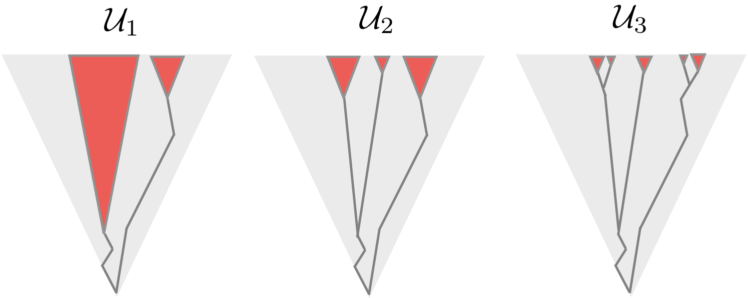
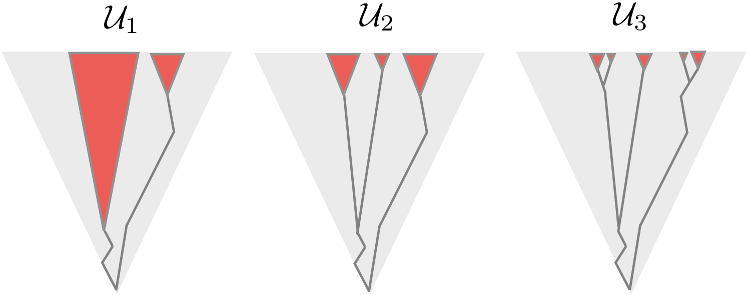


$$\sum_{\sigma \in U_1} 2^{-|\sigma|} \leq \frac{1}{2}$$



$$\sum_{\sigma \in U_1} 2^{-|\sigma|} \leq \frac{1}{2} \qquad \sum_{\sigma \in U_2} 2^{-|\sigma|} \leq \frac{1}{4}$$

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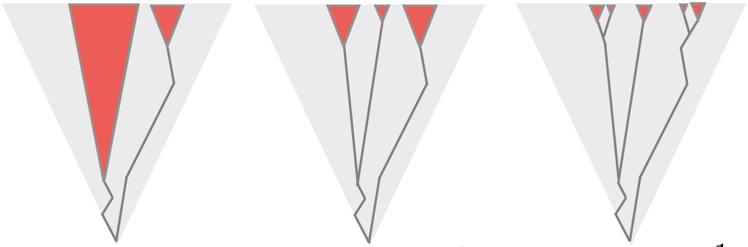

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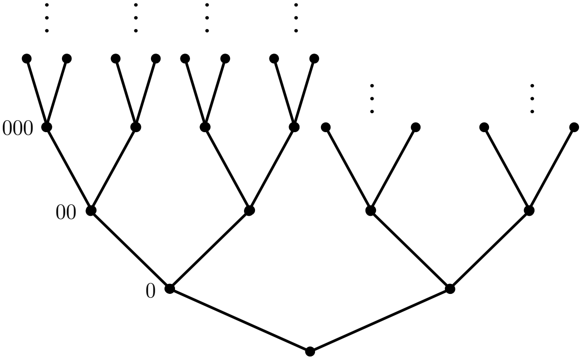
$$\sum_{\sigma \in U_1} 2^{-|\sigma|} \leq \frac{1}{2} \qquad \sum_{\sigma \in U_2} 2^{-|\sigma|} \leq \frac{1}{4} \qquad \sum_{\sigma \in U_3} 2^{-|\sigma|} \leq \frac{1}{8}$$



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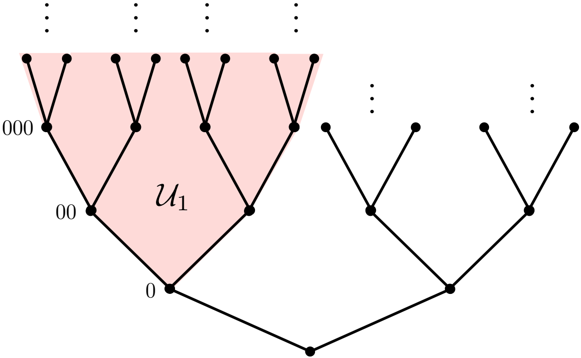
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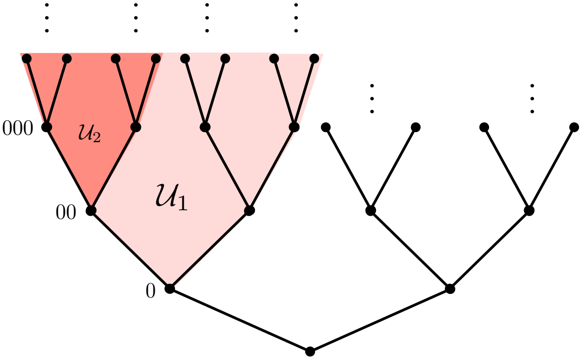
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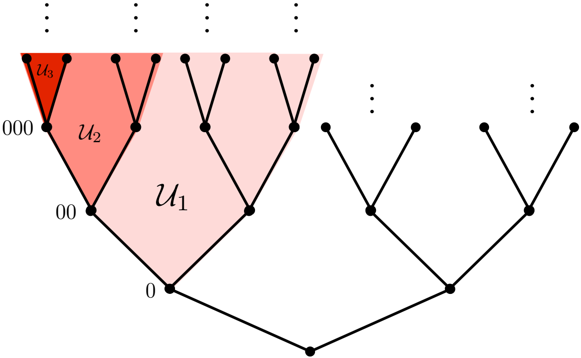

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2. The Levin-Schnorr theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ *is Martin-Löf random if and only if*

$$\forall n\, K(X \restriction n) \geq n - O(1).$$

Proof idea

For one direction, the strategy is to show that the compressible sequences c-compressible strings for various $c \in \mathbb{N}$ can be used to define a Martin-Löf test.

For the other direction, the strategy is to show that for each Martin-Löf test, there is some machine that compresses those sequences that do not pass the test.

Martin-Löf random $\Rightarrow$ Kolmogorov incompressible

Suppose that X is not Kolmogorov incompressible; that is, for every i, there is some n_i such that

$$K(X \upharpoonright n_i) < n_i - i.$$

Let $U_i = \{\sigma : K(\sigma) < |\sigma| - i\}$. Then

$$\sum_{\sigma \in U_i} 2^{-|\sigma|} \leq \sum_{\sigma \in U_i} 2^{-K(\sigma)-i} \leq 2^{-i}.$$

Setting $\mathcal{U}_i = \bigcup_{\sigma \in U_i} [\![\sigma]\!]$, it follows that $(\mathcal{U}_i)_{i \in \omega}$ is a Martin-Löf test containing X.

Kolmogorov incompressible $\Rightarrow$ Martin-Löf random

Suppose that $X \in \bigcap_{i \in \omega} \mathcal{U}_i$ for some Martin-Löf test $(\mathcal{U}_i)_{i \in \omega}$.

Idea: Build a prefix-free machine M such that if σ determines an open subset of $\mathcal{U}_{2i}$, then we set $M(\tau) = \sigma$ for some τ with $|\tau| \leq |\sigma| - i$.

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Kraft's inequality: If $\sum_{i\in\omega} 2^{-n_i} \leq 1$, then there is an instantaneous code consisting of codewords with lengths in $(n_i)_{i\in\omega}$.

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Effective version of Kraft's inequality: Given an effective list of pairs (σ_i, n_i) such that $\sum_{i\in\omega} 2^{-n_i} \leq 1$, there is a prefix-free machine M such that $K_M(\sigma_i) \leq n_i$.

3. Randomness with respect to a computable measure

Computable measures

We can also define Martin-Löf randomness with respect to any computable measure on 2^ω.

Definition

A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu(\llbracket\sigma\rrbracket)$ is computable as a real-valued function.

In other words, μ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2 \cap [0, 1]$ such that

$$|\mu(\llbracket\sigma\rrbracket) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$. (Here $\mathbb{Q}_2 = \{\frac{m}{2^n} : m, n \in \omega\}$.)

From now on we will write $\mu(\sigma)$ instead of $\mu(\llbracket\sigma\rrbracket)$.

MLR with respect to a computable measure

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes every μ-Martin-Löf test.

Hereafter, we will refer to a sequence as *proper* if it is random with respect to some computable measure.

Atomic computable measures

A measure μ is *atomic* if there is some $X \in 2^\omega$ such that $\mu(\{X\}) > 0$; otherwise μ is *continuous*.

Note that if X is an atom of a computable measure μ, then $X \in \mathrm{MLR}_\mu$.

Every computable sequence is the atom of some computable measure, namely the Dirac measure δ_X that concentrates all of its measure on X.

In fact, the converse holds: if X is the atom of a computable measure, then X is a computable sequence.

Generalizing the Levin-Schnorr Theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ is Martin-Löf random if and only if

$$\forall n\, K(X \restriction n) \geq n - O(1).$$

Generalizing the Levin-Schnorr Theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ is Martin-Löf random if and only if

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Theorem

Let μ be a computable measure on 2^ω. Then $X \in 2^\omega$ is μ-Martin-Löf random if and only if

$$\forall n\; K(X\restriction n) \geq -\log(\mu(X\restriction n)) - O(1).$$

4. The initial segment complexity of proper sequences

Complex sequences

An *order function* $h : \omega \to \omega$ is an unbounded, non-decreasing function.

Definition

$X \in 2^\omega$ is *complex* if there is a computable order function $h : \omega \to \omega$ such that

$$\forall n\, K(X \upharpoonright n) \geq h(n).$$

Proper sequences and complexity

Suppose that X is Martin-Löf random with respect to a computable measure μ.

Then by the generalized version of the Levin-Schnorr theorem,

$$\forall n \; K(X \upharpoonright n) \geq -\log(\mu(X \upharpoonright n)) - O(1).$$

Note that this does not imply that X is complex, since the function $n \mapsto -\log(\mu(X \upharpoonright n))$ is in most cases not computable but only X-computable.

Are there conditions that guarantee that a proper sequence is complex?

A priori complexity

Definition

- A *semi-measure* is a function $\rho : 2^{<\omega} \to [0,1]$ satisfying
  (i) $\rho(\epsilon) = 1$ and
  (ii) $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.
- A semi-measure ρ is *left-c.e.* if ρ is computably approximable from below.

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Fact: There exists a *universal* left-c.e. semi-measure M. That is, for every left-c.e. semi-measure ρ there is some c such that

$$c \cdot M(\sigma) \geq \rho(\sigma)$$

for every σ.

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for every σ.

We define the *a priori complexity* of $\sigma \in 2^{<\omega}$ to be

$$KA(\sigma) := -\log M(\sigma).$$

A sufficient condition for complexity

Theorem (Hölzl, Merkle, Porter)

If $X \in 2^\omega$ is Martin-Löf random with respect to a computable, continuous measure μ, then X is complex.

A sufficient condition for complexity

Theorem (Hölzl, Merkle, Porter)

If $X \in 2^\omega$ is Martin-Löf random with respect to a computable, continuous measure μ, then X is complex.

This follows from the following two results.

- Let μ be a computable, continuous measure and let $X \in \mathrm{MLR}_\mu$. Then X computes some $Y \in \mathrm{MLR}$ by an effective procedure that is total on all oracles.
- If Y is complex and X computes Y by an effective procedure that is total on all oracles, then X is complex.

What about the converse?

The converse of the previous theorem doesn't hold, as there are complex sequences that are not proper.

However, we do have a partial converse.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

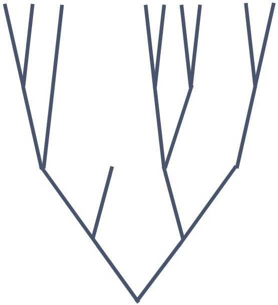
A useful lemma

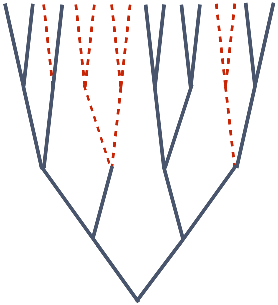
Lemma

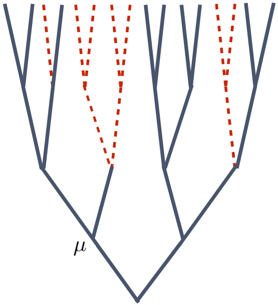
Suppose that

- μ *is a computable measure,*
- $X \in \mathrm{MLR}_\mu$ *is non-computable,*
- $\mathcal{P}$ *is a* Π^0_1 *class with no computable members, and*
- $X \in \mathcal{P}$.

Then there is some computable, continuous measure ν *such that* $X \in \mathrm{MLR}_\nu$.



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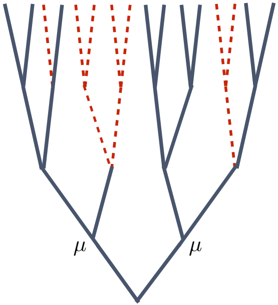
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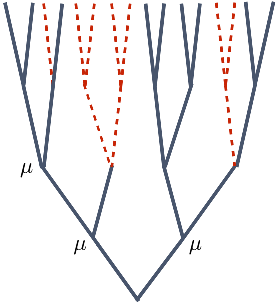
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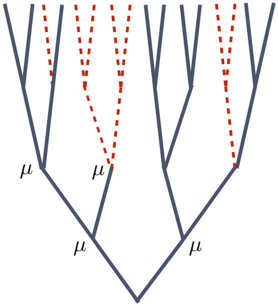
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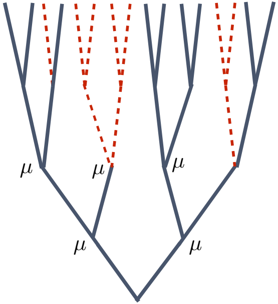

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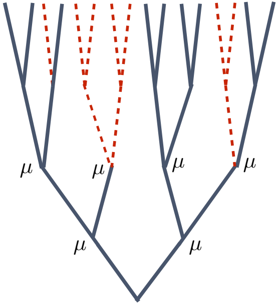

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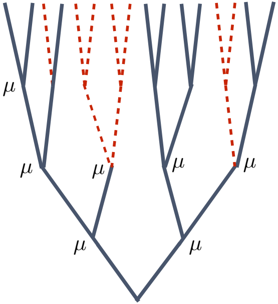

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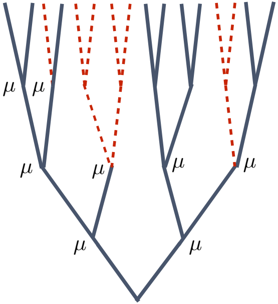

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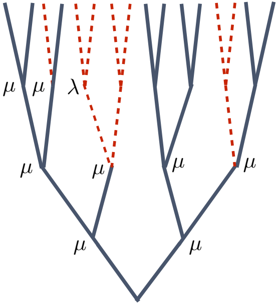

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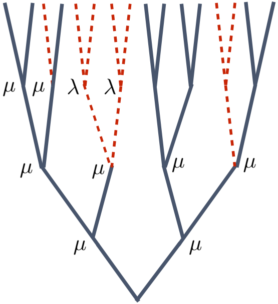
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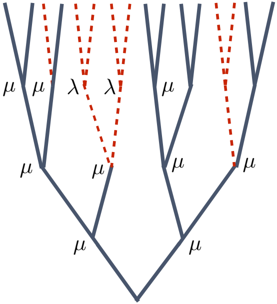
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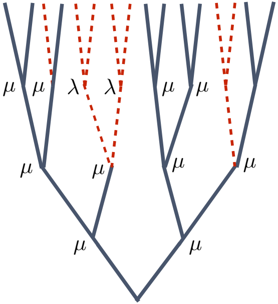
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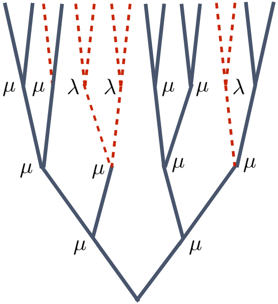
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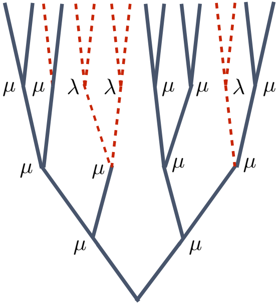

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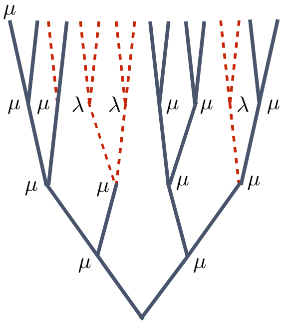
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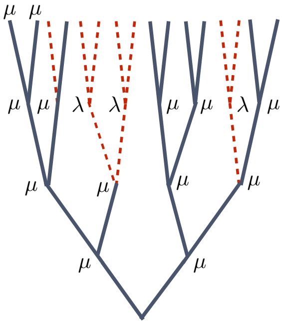
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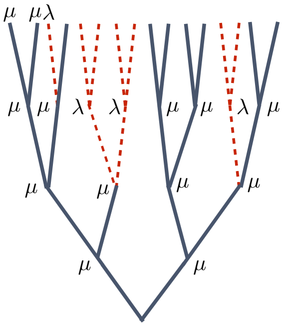

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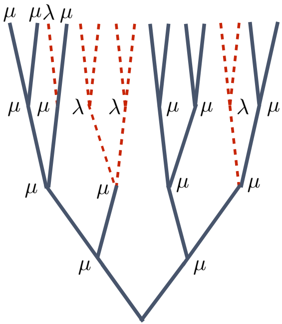
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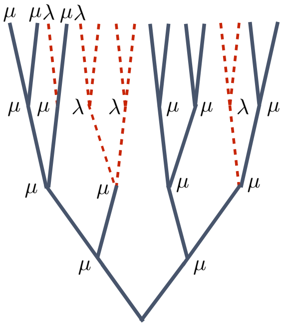
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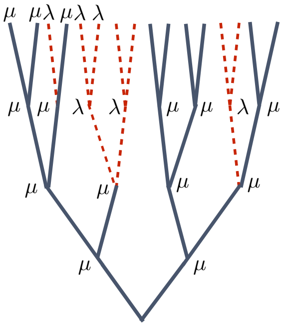

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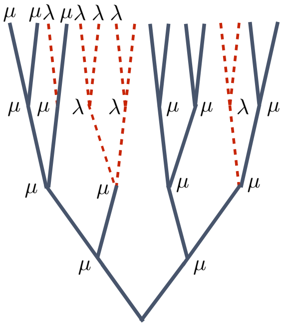
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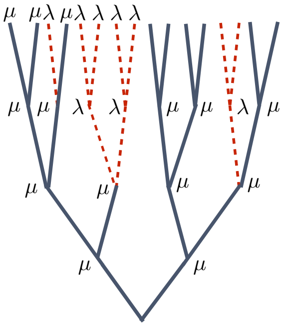
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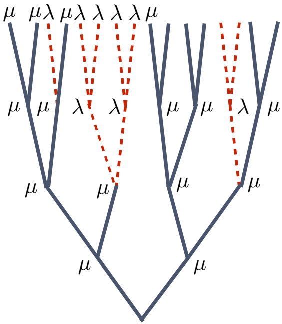
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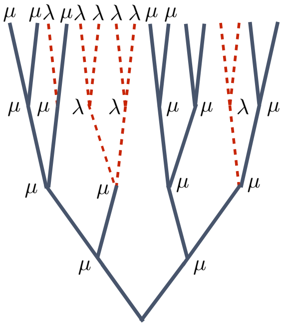
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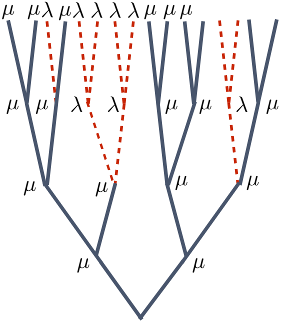
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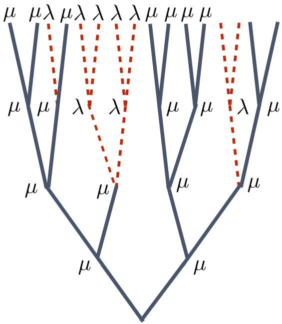

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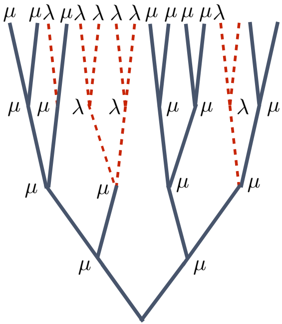
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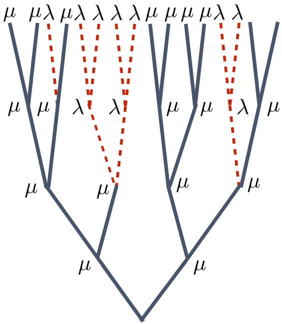
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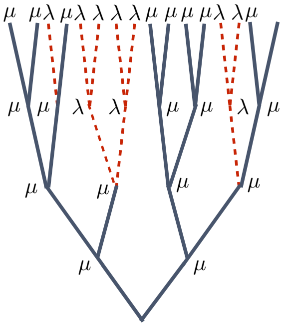
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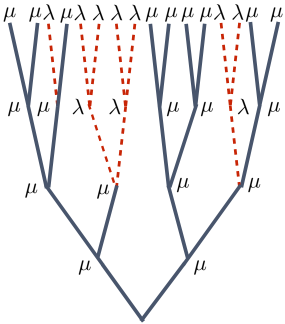
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Establishing the partial converse

Theorem

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

Establishing the partial converse

Theorem

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

To prove this theorem, let h be the computable order function that witnesses that X is complex.

Then we apply the previous lemma to the Π^0_1 class

$$\{A \in 2^\omega : K(A{\upharpoonright}n) \geq h(n)\},$$

which contains X but no computable sequences.

Connection to semigenericity

Definition

$X \in 2^\omega$ is *semigeneric* if X is non-computable and for every Π^0_1 class $\mathcal{P}$ with $X \in \mathcal{P}$, $\mathcal{P}$ contains some computable member.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. *$X \in \mathrm{MLR}_\mu$ for some computable, continuous μ.*
2. *X is complex.*
3. *X is not semigeneric.*

A follow-up question

Let μ be a computable, continuous measure.

Since every sequence that is random with respect μ is complex, is there a single computable order function that witnesses the complexity of μ-random sequences?

Is there a least such function (up to an additive constant)?

A follow-up result

Definition

Let μ be a continuous measure. Then the *granularity function of μ*, denoted g_μ, is the order function mapping n to the least ℓ such that $\mu(\sigma) < 2^{-n}$ for every σ of length ℓ.

A follow-up result

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Theorem (Hölzl, Merkle, Porter)

Let μ be a computable, continuous measure and let $X \in \mathrm{MLR}_\mu$. Then we have

$$\forall n\, KA(X \upharpoonright n) \geq g_\mu^{-1}(n) - O(1).$$

Some facts about the granularity of a computable measure

- If μ is exactly computable, that is, μ is $\mathbb{Q}_2$-valued and the function $\sigma \mapsto \mu(\sigma)$ is a computable function, then g_μ is computable.

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Some facts about the granularity of a computable measure

- If μ is exactly computable, that is, μ is $\mathbb{Q}_2$-valued and the function $\sigma \mapsto \mu(\sigma)$ is a computable function, then g_μ is computable.
- However, there is a computable, continuous measure μ such that the granularity function g_μ of μ is not computable.
- For every computable, continuous measure μ, there is a computable order function $f : \omega \to \omega$ such that

$$|f(n) - g_\mu(n)^{-1}| \leq O(1).$$

Such a function f provides as a global computable lower bound for the initial segment complexity of every μ-random sequence.

A question about uniformity

Question

If we have a computable, atomic measure μ such that

$$\forall X \in 2^\omega \ (X \in \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \ \Rightarrow \ X \text{ is complex}),$$

is there a computable, continuous measure ν such that

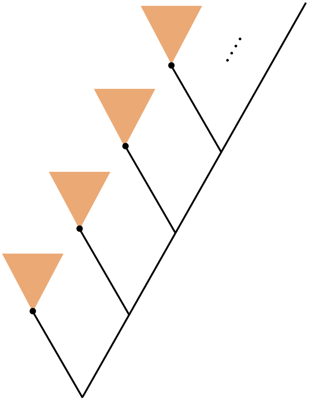
$$\mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \subseteq \mathrm{MLR}_\nu?$$

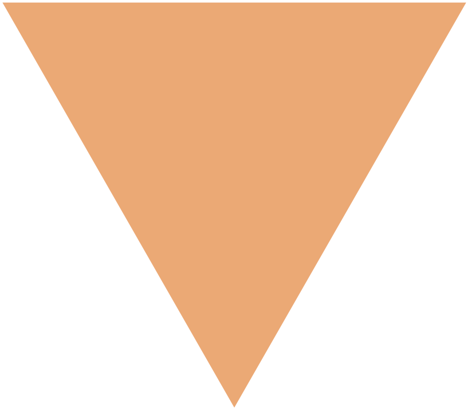
An answer

Theorem (Hölzl, Merkle, Porter)

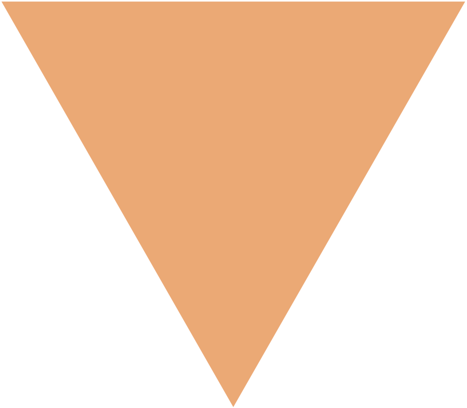
There is a computable, atomic measure μ *such that*

- *every* $X \in \mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu$ *is complex but*
- *there is no computable, continuous measure* ν *such that* $\mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \subseteq \mathsf{MLR}_\nu$.

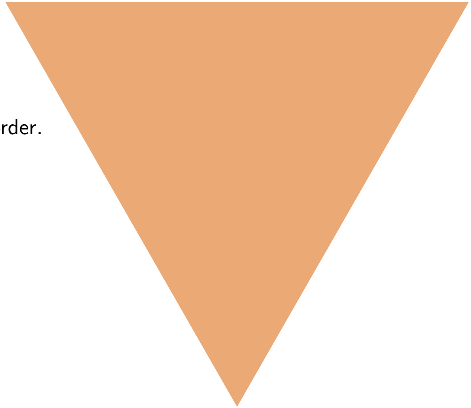




the i^{th} neighborhood

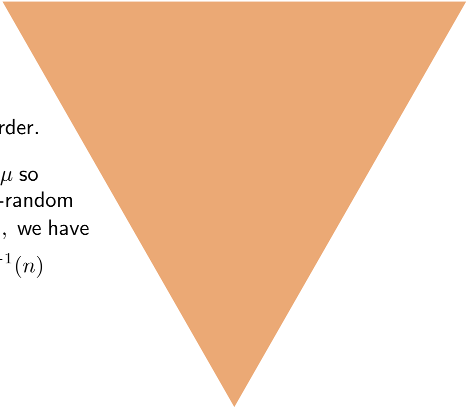


the i^{th} neighborhood



Suppose that ϕ_i is an order.

the i^{th} neighborhood



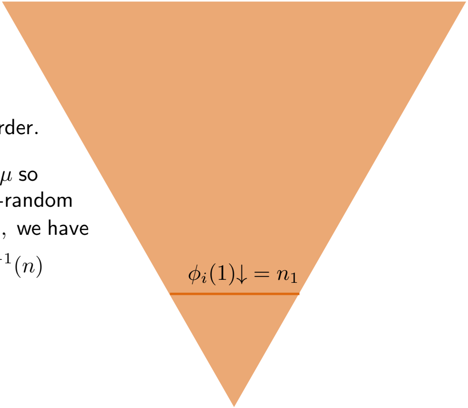
Suppose that ϕ_i is an order.

We define the measure μ so that for any complex μ-random X in this neighborhood, we have

$$KA(X{\upharpoonright}n) < \phi_i^{-1}(n)$$

for almost every n.

the i^{th} neighborhood



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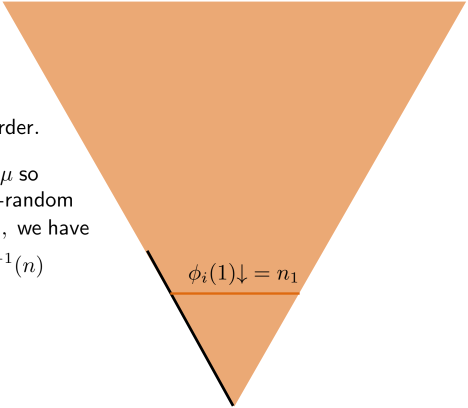

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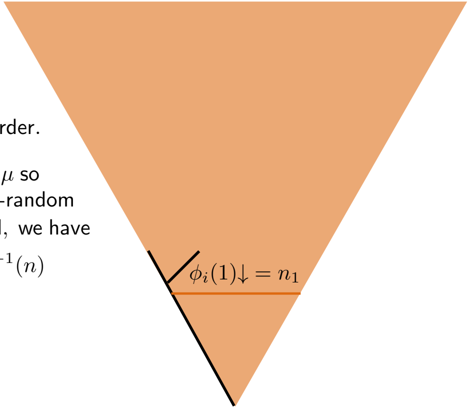

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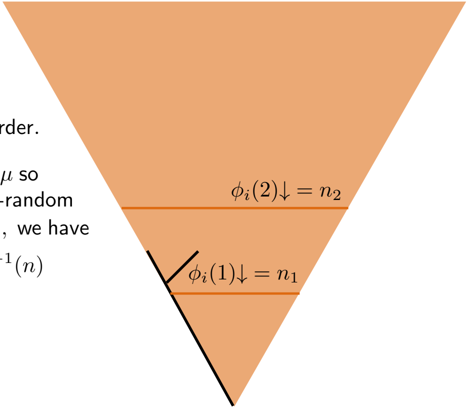

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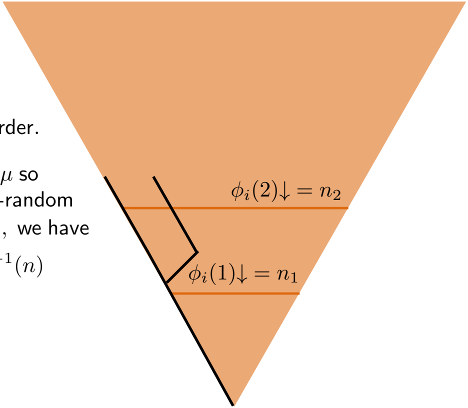

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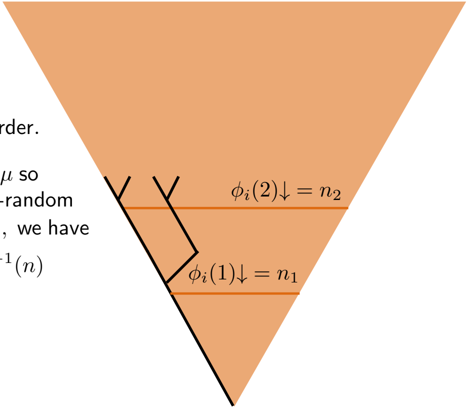

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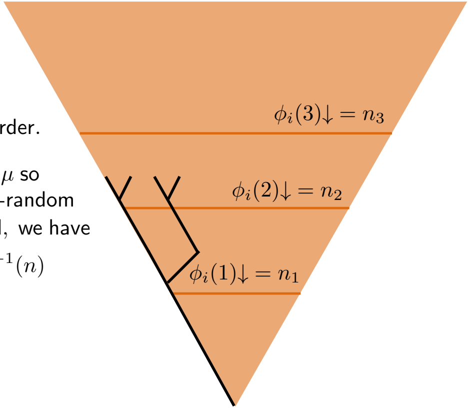

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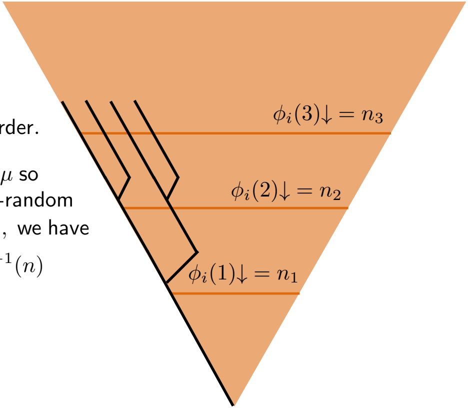

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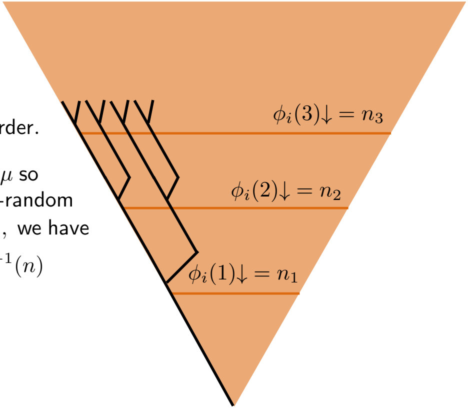

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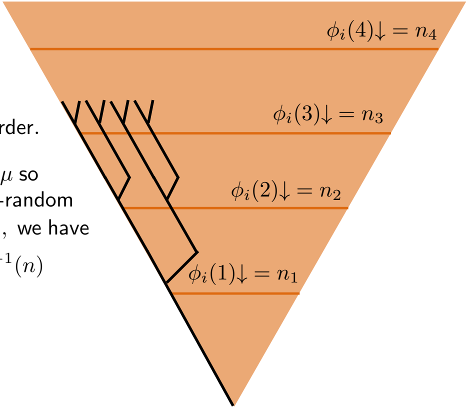

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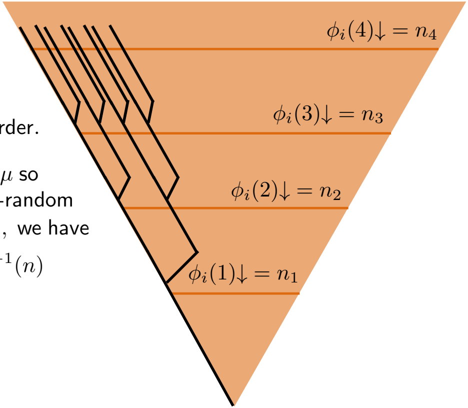

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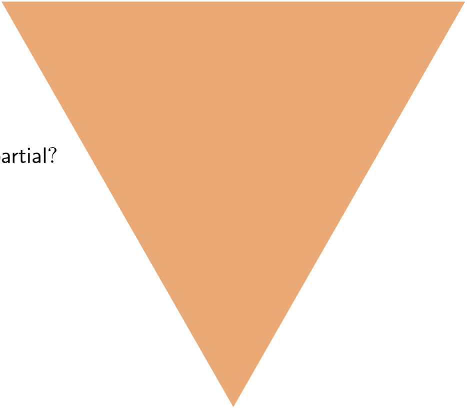
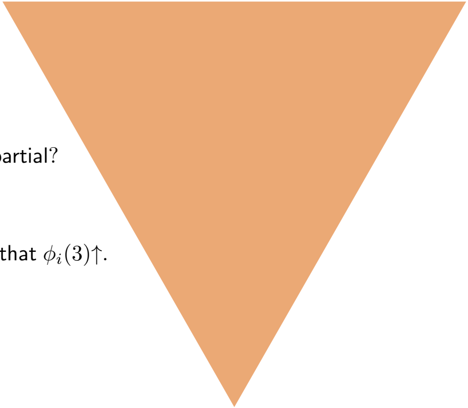


What happens if ϕ_i is partial?

the i^{th} neighborhood



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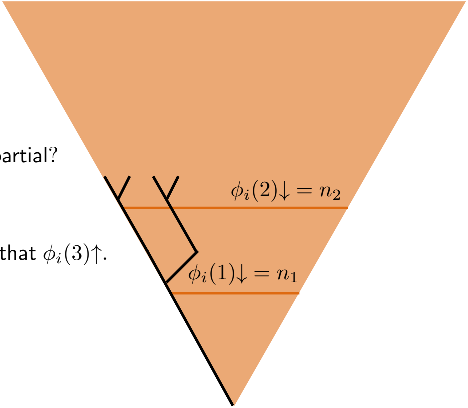
Suppose, for instance, that $\phi_i(3)\uparrow$.

the i^{th} neighborhood



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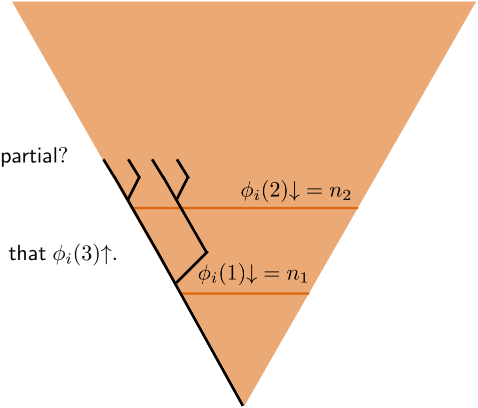

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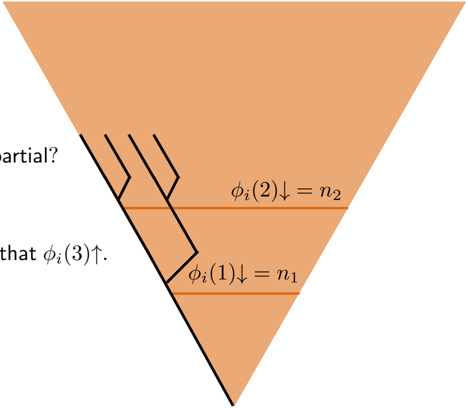

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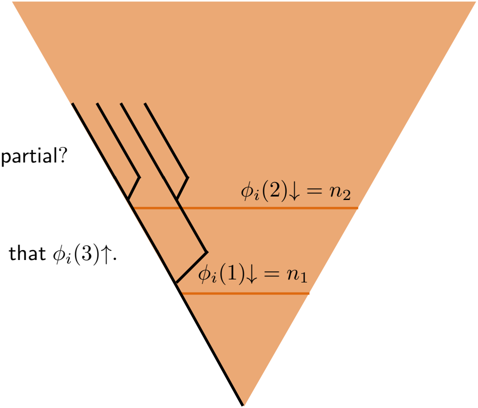

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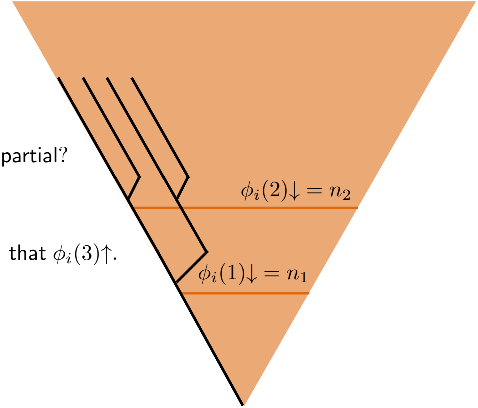

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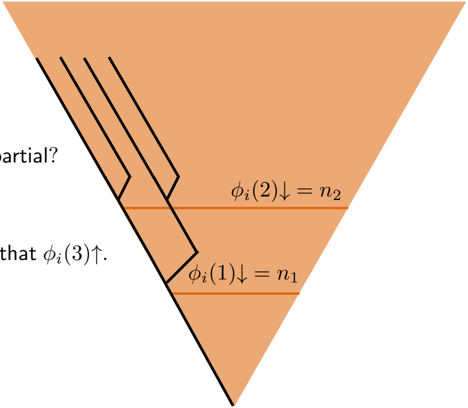

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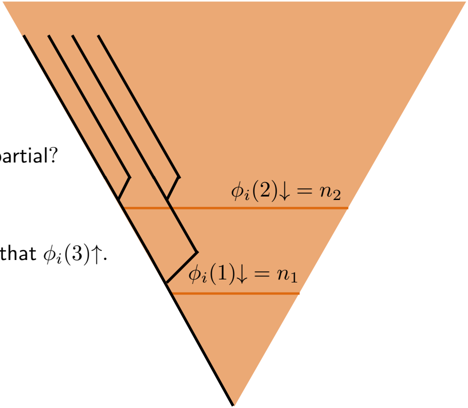

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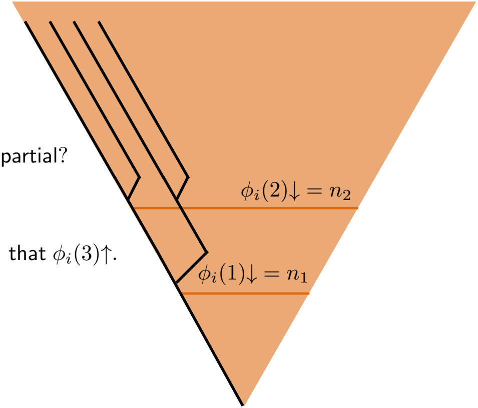

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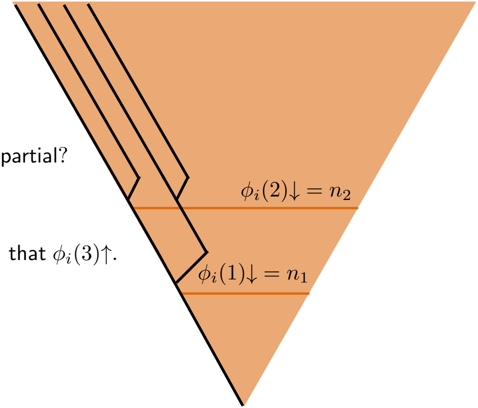

Let $\llbracket \sigma_i \rrbracket$ be the i^{th} neighborhood.

One can verify that

- if ϕ_i is partial, then $\llbracket \sigma_i \rrbracket \cap \mathsf{MLR}_\mu \subseteq \mathsf{Atoms}_\mu$;
- if ϕ_i is total, then $\llbracket \sigma_i \rrbracket \cap \mathsf{Atoms}_\mu = \emptyset$ and every $X \in \mathsf{MLR}_\mu \cap \llbracket \sigma_i \rrbracket$ is complex.

Lastly, if there is some computable, continuous ν such that $\mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \subseteq \mathsf{MLR}_\nu$, then there is a computable order $f = \phi_i$ such that for every $X \in \mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu$,

$$KA(X{\upharpoonright}n) \geq f^{-1}(n) - O(1)$$

for every n, which yields a contradiction.

Thank you!