The Logical Approach to Randomness

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Introduction

The concept of randomness plays an important role in mathematical practice, particularly in areas such as

- probability theory,
- mathematical statistics,
- real analysis,
- dynamical systems,
- combinatorics, and
- number theory.

Although the forms that randomness takes can vary across mathematical disciplines, one standard definition of randomness is to take an object to be random if it is obtained as the result of randomly choosing an element from some fixed collection of objects.

An alternative approach

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Previous philosophical treatments of the theory of algorithmic randomness have not explicitly presented it as an instance of a more general logical approach to defining randomness, nor have they attended to certain problems that arise when one defines randomness in such a general manner.

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2. allows us to identify and diagnose the source of certain conceptual challenges that face a large family of formal definitions of randomness; and
3. takes into account recent work on algorithmic randomness in various spaces with respect to various probability measures.

Outline

1. Logical definitions of randomness
2. Two stability problems for the logical approach
3. Towards a solution to the stability problems

1. Logical definitions of randomness

The main ingredients of a logical definition

Each logical definition of randomness is formulated in terms of three ingredients:

- a collection $\mathscr{X}$ of objects (usually a metric space with the Borel σ-algebra);
- a probability measure μ on $\mathscr{X}$; and
- a collection of properties $\{\Phi_i(x)\}_{i\in\omega}$, expressible in some formal language $\mathcal{L}$, and satisfiable by objects in $\mathscr{X}$, such that for each $i \in \omega$,

$$\mu(\{x \in \mathscr{X} : \Phi_i(x)\}) = 1.$$

Hereafter, I will refer to the properties $\{\Phi_i\}_{i\in\omega}$ as *randomness properties*.

Putting the ingredients together

From a triple $(\mathscr{X}, \mu, \{\Phi_i\}_{i\in\omega})$ satisfying the conditions from the previous slide, we get a definition $\mathscr{D}$ of μ-randomness for objects in $\mathscr{X}$ by stipulating that

$$x \in \mathscr{X} \text{ is } \mathscr{D}\text{-random if and only if } \Phi_i(x) \text{ for every } i \in \omega.$$

One immediate consequence of this definitional framework is that, assuming that there is some $x \in \mathscr{X}$ and some $i \in \omega$ such that x does not satisfy Φ_i, we can partition $\mathscr{X}$ into

- a non-empty collection of $\mathscr{D}$-random objects, and
- a non-empty collection of non-$\mathscr{D}$-random objects.

Examples of logical definitions of randomness (1)

Let $(\mathscr{X}, \mu)$ be a probability space that satisfies certain effectivity conditions, where μ be a probability measure whose values on basic open subsets of $\mathscr{X}$ can be effectively approximated. For example:

- $[0, 1]$ with the Lebesgue measure;
- 2^{ω} with a Bernoulli measure with parameter $p \in \mathbb{Q}$;
- $\mathcal{C}[0, 1]$ with the Wiener measure.

On such spaces, there are a number of non-equivalent definitions of algorithmic randomness.

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In general, these definitions are not equivalent.

Valuative randomness

The logical approach to randomness differs radically from one frequently occurring notion of randomness in classical mathematics, which I refer to as *valuative randomness*.

Roughly speaking, the idea behind valuative randomness is this: to be random is to be the value of a random variable.

Recall that a random variable is a measurable function from a sample space Ω to some space, usually $\mathbb{R}$.

The usage of 'random' is not exact here; randomness is usually attributed to the function itself, but sometimes it is also attributed to individual outputs of the function.

ϕ-valued random variables

However, it is important to emphasize that in practice, the range of a random variable can be any collection of mathematical objects:

- complex numbers
- vectors
- matrices
- functions
- graphs
- closed sets
- measures
- and so on...

Let ϕ be a mathematical object such as one from any of the collections listed above.

Then a random ϕ is simply a ϕ-valued random variable.

Where exactly is the randomness?

It is common to think of a random variable as yielding the values of some random or chancy experiment (such as some measurement of some randomly selected individual).

Thus, a ϕ-valued random variable can be understood as yielding as output a randomly chosen ϕ from the relevant collection of objects.

Note that this random experiment/choice isn't technically part of the definition of a random variable, but in applications, such experiments or choices are often associated to random variables.

Almost sure events

Random variables can take values that we would not expect to arise as the result of some random experiment.

For instance, a real-valued random variable can take the value $0.1111111\ldots$, or a graph-valued random variable can produce a complete graph as output.

However, there is a sense in which such outcomes are atypical.

In particular, one can associate a probability distribution to a random variable, and by means of such a probability distribution, one can define events that happen almost surely (i.e. with probability one).

Thus, if some property Θ occurs almost surely with respect to the probability distribution associated to a ϕ-valued random variable, we say, "a random ϕ has Θ almost surely."

Comparing the logical and valuative approaches

The key distinction between the logical and valuative approaches is the former is *discriminative* while the latter is not.

That is, on the logical approach, one discriminates between the random and the non-random objects.

By contrast, on the valuative approach, *any* object in the relevant domain of objects can be the value of a random variable (and thus can be counted as random).

Moreover, on the valuative approach, one does not typically attribute non-randomness to any objects.

Process randomness vs. product randomness

It is worth noting that a similar distinction appears in the philosophical literature on algorithmic randomness, namely the distinction between *process randomness* and *product randomness*.

- An object is process random if it is produced by a random process.
- An object is product random if it bears those properties that are typically held by the products of a random process.

For reasons we can discuss later, this distinction is less helpful than the distinction between the logical approach and the valuative approach to randomness for the purposes of discussing the uses of randomness in mathematical practice.

2. Two stability problems for the logical approach

The flavor of the stability problems

The logical approach to defining randomness faces two serious problems, which I refer to as

- the randomness property problem; and
- the underlying measure problem.

The general thrust of these problems is that each logical definition of randomness depends on the choice of specific parameters, which, if not chosen on some principled basis, threaten to trivialize the logical approach to randomness.

Given that logical definitions of randomness appear to be vulnerable to slight perturbations of these parameters, I refer to these problems as *stability problems*.

Motivating the randomness property problem

For each object $x \in \mathscr{X}$, if we let the formula $\phi_x(y)$ be

$$y \neq x,$$

then assuming that $\mu(\{x\}) = 0$, we will have

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Moreover, if μ is continuous (i.e., $\mu\big(\{y\}\big) = 0$ for every $y \in \mathscr{X}$), then each of the formulas in $\{\phi_x\}_{x \in \mathscr{X}}$ defines a set of μ-measure one.

The randomness property problem (1)

The example on the previous slide shows that one cannot require random objects to satisfy *every* measure one property, for otherwise the resulting definition of randomness would have an empty extension.

But note that for any given object $x \in \mathscr{X}$, we can always include the property ϕ_x among the collection $\{\Phi_i\}_{i\in\omega}$ of randomness properties.

That is, for any $x \in \mathscr{X}$ there is always some choice of randomness properties that excludes x as non-random.

In light of this problem, for nearly 45 years, one central question in the development of algorithmic randomness was: Which properties should we count as *the* randomness properties?

The randomness property problem (2)

The answer to this question about a choice of randomness properties depends on the role we want a logical definition of randomness to play.

We can thus cast the randomness property problem relative to some aims or purposes:

RPP: For a given set of purposes, is there a principled choice of measure one properties as the randomness properties that yields a notion of randomness that successfully fulfills these purposes?

The RPP in context (1)

The prototype for the logical definitions that are studied today was first given by von Mises in 1919.

On von Mises' approach, a sequence is random if

(i) the limiting relative frequency of each element in the sequence exists, and

(ii) this limiting relative frequency is invariant under selecting subsequences from the original sequence.

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(ii) this limiting relative frequency is invariant under selecting subsequences from the original sequence.

Note that we cannot require invariance under the selection of all possible subsequences, as the only sequences that would be counted as random are those that are nearly constant.

However, von Mises did not initially specify which selection rules were to be used in his definition.

Aware of this problem, von Mises' contemporaries objected that his definition was defective.

The RPP in context (2)

Subsequently, Wald proved that any *countable* collection of selection rules yields a definition of randomness satisfied by continuum many sequences.

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- Wald: Those rules definable in some "logic."
- Church: Those rules that are effectively computable.
- Kruse: Studied selection rules definable in various set theories.
- Agafonov: Rules given by a finite-state automaton yield the normal sequences as random.

The RPP in context (3)

Ville proved that no matter which countable collection of selection rules is chosen, there is some measure one property that fails to be satisfied by the resulting notion of randomness.

In response to this problem, Ville developed his own notion of randomness in terms of certain betting strategies he called *martingales*.

Ville proved that the collection of sequences on which a martingale fails to win unbounded capital has measure one (again, with respect to the relevant measure).

Which martingales should be used to define randomness?

The RPP in context (4)

Martin-Löf: Martin-Löf tests capture all randomness properties that one will encounter in “present or future use in statistics.”

Schnorr argued that Martin-Löf tests yield too many randomness properties and thus fail to capture “the true concept of randomness.”

In Schnorr’s view, only measure one properties defined by Martin-Löf tests that are “visualizable” should be counted as randomness properties.

Such properties correspond precisely to the collection of Schnorr tests.

Summing up

Despite these latter developments, over the 40+ years since the contributions of Martin-Löf and Schnorr, there has yet to be a clear articulation of what these definitions are intended to capture.

That being the case, no one has offered a systematic account as to why any of the currently available definitions of algorithmic randomness adequately address the randomness property problem.

Motivating the underlying measure problem

For a given probability space $\mathscr{X}$, a formula ϕ that defines a set of measure one with respect to one measure μ may define a set of measure zero with respect to another measure ν (so that the formula $\neg\phi$ defines a set of ν-measure one).

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Consequently, the extension of any logical definition of randomness with underlying measure μ that counts the property ϕ as a randomness property will be disjoint from the extension of any logical definition with underlying measure ν that counts the property $\neg\phi$ as a randomness property.

To complicate matters...

For any object $x \in \mathscr{X}$, there is a measure μ_x on $\mathscr{X}$ such that $\mu_x(\{x\}) = 1$ (i.e., the *Dirac measure* concentrated on x).

Then the only μ-random element of $\mathscr{X}$ is x.

Thus, if we are too general in our approach to defining randomness, we run the risk of counting every object as random with respect to some definition.

Randomness with respect to non-computable measures

One need not appeal to Dirac measures to formulate the underlying measure problem.

If we consider, say, Martin-Löf randomness with respect to non-computable measures on 2^ω, one can prove the following:

Theorem (Reimann-Slaman)

For every sequence $X \in 2^\omega$, X is non-computable if and only if there is some measure μ such that

(i) $\mu(\{X\}) = 0$ *and*

(ii) X *is Martin-Löf random with respect to* μ.

Surprisingly, this fact can be witnessed by a *single* measure!

The underlying measure problem

UMP: How can we countenance notions of randomness with respect to different probability measures without potentially counting every object as random?

More concisely, which measures yield “legitimate” notions of randomness?

3. Towards a solution of the stability problems

One possible strategy

One strategy for responding to these problems is to identify a definition of randomness, given by one collection of randomness properties and one underlying measure, and successfully argue that this is the correct definition.

Just as the notion of Turing computable function captures the intuitive conception of effectively calculable function, we could hope to isolate a single definition of randomness that captures the intuitive conception of randomness.

A worry about this strategy

Although some have held that there is such a single correct definition of randomness, this view has always been articulated for definitions of random sequence with respect to the Lebesgue measure.

For instance, both Martin-Löf randomness and Schnorr randomness have been held to capture the intuitive conception of randomness.

But what about definitions of randomness for other objects, and with respect to different measures?

Should we hope for one general definition of randomness that is correct for each choice of objects and each choice of underlying measure?

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2. Concede that logical definitions of randomness give rise to artifacts, i.e. unintended consequences that result when we apply the tools of logic to the task of defining randomness.
    - Classify the various kinds of artifacts that arise.
    - Diagnose the sources of these artifacts.

Almost sure behavior

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More significantly, we have a number of stronger results of the form:

ϕ has Θ *if and only if* ϕ is algorithmically random.

A theorem involving almost sure behavior

Consider the following example:

Theorem: For every real-valued function $f : [0,1] \to \mathbb{R}$ of bounded variation, f is differentiable almost everywhere.

A few observations:

- The function quantifier in this theorem ranges over sets of size $2^{\mathfrak{c}}$, the size of the power set of the continuum.
- The properties "being a point of differentiability of some real-valued function of bounded variation" and "being a point of non-differentiability of some real-valued function of bounded variation" are satisfied by every point in [0,1].

A restricted version of the theorem

Now consider:

For every *computable* non-decreasing real-valued function $f : [0,1] \to \mathbb{R}$, f is differentiable almost everywhere.

A few observations:

- The function quantifier in this theorem now ranges over countably many functions.
- Thus the property "being a point of differentiability of every computable real-valued function of bounded variation" is the intersection of countably many sets of Lebesgue measure one, which is itself a set of Lebesgue measure one.

The connection to randomness

Theorem (Brattka, Miller, Nies)

$z \in [0,1]$ is Martin-Löf random if and only if every computable, real-valued function $f : [0,1] \to \mathbb{R}$ of bounded variation is differentiable at z.

That is, Martin-Löf randomness is necessary and sufficient for this particular instance of almost sure behavior.

Almost sure behavior in classical analysis

Such results hold for a number of definitions of algorithmic randomness:

$x \in \mathrm{MLR}$	$\Leftrightarrow$	Every computable real-valued function of bounded variation is differentiable at x.
$x \in \mathrm{SR}$	$\Leftrightarrow$	For every L_1-computable real-valued function f, the Lebesgue differentiation theorem holds for f at x.
$x \in \mathrm{W2R}$	$\Leftrightarrow$	Every computable real-valued a.e.-differentiable function is differentiable at x.

There are a number of other examples, some involving definitions of randomness that we have not considered here.

Almost sure behavior in ergodic theory

$x \in \mathrm{MLR}_\mu$	$\Leftrightarrow$	Birkhoff's ergodic theorem holds at x for all computable ergodic transformations with respect to every lower semi-computable function.
$x \in \mathrm{SR}_\mu$	$\Leftrightarrow$	Birkhoff's ergodic theorem holds at x for all computable ergodic transformations with respect to every computable function.
$x \in \mathrm{W2R}_\mu$	$\Rightarrow$	A weak version of Birkhoff's ergodic theorem holds at x for all computable measure-preserving transformations with respect to every lower semi-computable function.

More examples

There are other promising developments along similar lines:

- Martin-Löf random closed sets;
- Martin-Löf random Brownian motion;
- effective notions of Hausdorff and packing dimension.

What do these examples tell us?

Defintions such as MLR, SR, and W2R correspond to effective versions of almost sure behavior that are of independent interest to mathematics.

For the purposes of classifying the effective content of almost sure behavior in classical mathematics, these definitions thus prove to be extremely useful.

The different choices of randomness properties that yield these definitions are thus vindicated by these examples.

Artifacts of logical definitions

Although the previous results indicate that certain choices of randomness properties and underlying measures yield interesting and informative definitions of randomness, we still have to account for the pathological behavior that logical definitions can yield.

The challenge is determine which features of our logical definitions are artifacts and which are not.

The Reimann-Slaman example

The measures in Reimann-Slaman theorem are admittedly exotic (for instance, it is necessary that they give *some* points positive measure, i.e. they are necessarily discontinuous).

A case can be made that the Reimann-Slaman theorem and related results are artifacts of the computational framework used to define randomness (particularly when we consider non-computable measures).

But on what grounds can we rule out these definitions as illegitimate?

Restricting to computable measures?

The measures considered in mathematical practice are typically computable measures (the Lebesgue measure, Bernoulli measures with rational parameter p, etc.).

In fact, it is quite difficult to produce an example of a non-computable measure, especially without appealing to the standard tricks from computability theory.

The stability of randomness w.r.t. computable measures

Further, from the point of view of algorithmic randomness, there is a high degree of stability among the sequences random with respect to some computable measure:

Theorem (Levin-Kautz)

For every non-computable sequence X, if X is Martin-Löf random with respect to some computable measure μ, then X is Turing equivalent to a sequence Y that is Martin-Löf random with respect to the Lebesgue measure.

Does this stability justify a restriction to computable measures?

In conclusion

By grounding the choice of randomness properties and underlying measures in results concerning almost sure behavior in classical mathematics, we vindicate these choices.

However, much work remains to be done in accounting for which aspects of our logical definitions legitimately reflect features of mathematical randomness, and which are merely artifacts.