Deep Π^0_1 Classes

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Joint work with Laurent Bienvenu

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The opinions expressed in this talk are those of the speaker and do not necessarily reflect the views of the John Templeton Foundation.

Introduction

The goal of this talk is to discuss two sorts of Π^0_1 classes, each of which gives us insight into certain limitations of probabilistic computation:

1. negligible Π^0_1 classes;
2. deep Π^0_1 classes.

A brief history

Gödel's first incompleteness theorem tells us that there is no effective procedure for producing a consistent completion of Peano arithmetic (hereafter, PA).

In the early 1970's, Jockusch and Soare strengthened this result by proving (essentially) that the probability of producing a consistent completion of PA via a probabilistic procedure is zero.

In modern terminology, the set of consistent completions of PA is *negligible*.

A brief history, continued

In the early 2000s, Levin strengthened the Jockusch/Soare result by proving that the probability of producing some initial segment of a consistent completion of PA goes to zero *quickly*.

This property is what we have isolated as the notion of *depth*.

Outline of today's talk

1. Background
2. Negligible Π^0_1 classes
3. Deep Π^0_1 classes

1. Background

Notation

$2^{<\omega}$ is the collection of finite binary strings.

2^{ω} is the collection of infinite binary sequences.

The standard topology on 2^{ω} is given by the basic open sets

$$[\![\sigma]\!] = \{X \in 2^{\omega} : \sigma \prec X\},$$

where $\sigma \in 2^{<\omega}$ and $\sigma \prec X$ means that σ is an initial segment of X.

The Lebesgue measure on 2^{ω}, denoted λ, is defined by

$$\lambda([\![\sigma]\!]) = 2^{-|\sigma|}$$

for each $\sigma \in 2^{<\omega}$ (where $|\sigma|$ is the length of σ), and then we extend λ to all Borel sets in the usual way.

Martin-Löf Randomness

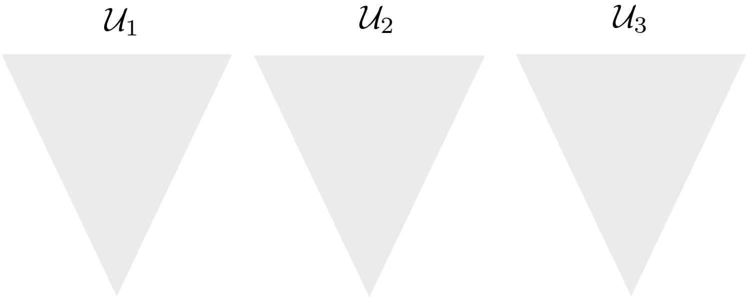
Definition

A *Martin-Löf test* is a uniformly Σ^0_1 sequence $(\mathcal{U}_i)_{i\in\omega}$ such that for each i,

$$\lambda(\mathcal{U}_i) \leq 2^{-i}.$$

A sequence $X \in 2^\omega$ *passes the Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

$X \in 2^\omega$ is *Martin-Löf random*, denoted $X \in \mathrm{MLR}$, if X passes *every* Martin-Löf test.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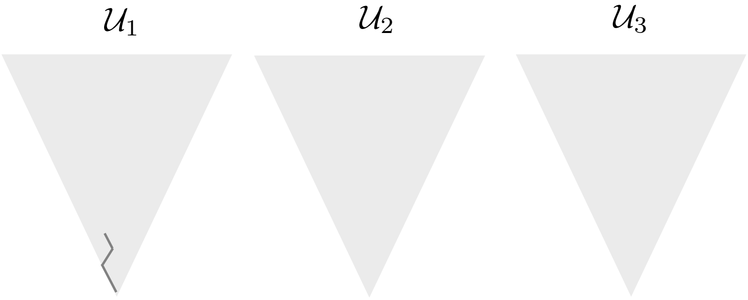
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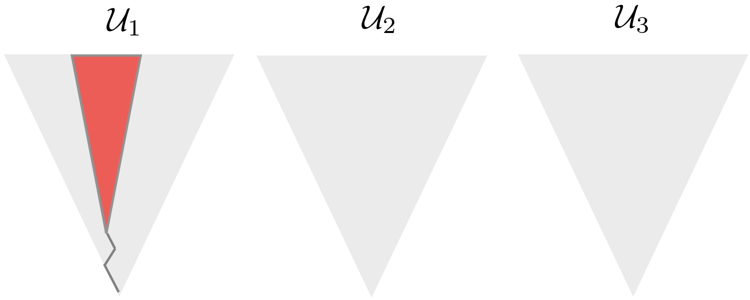
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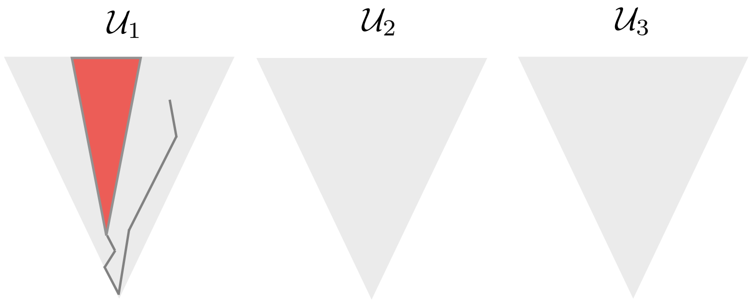
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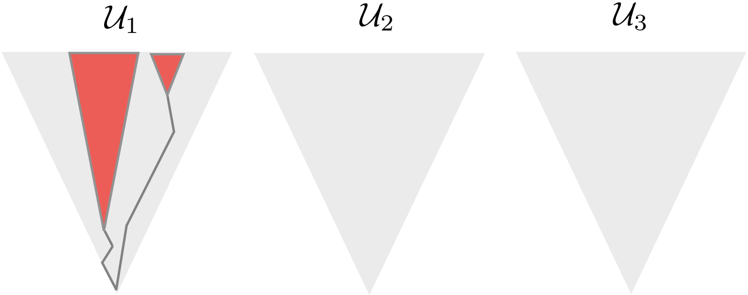
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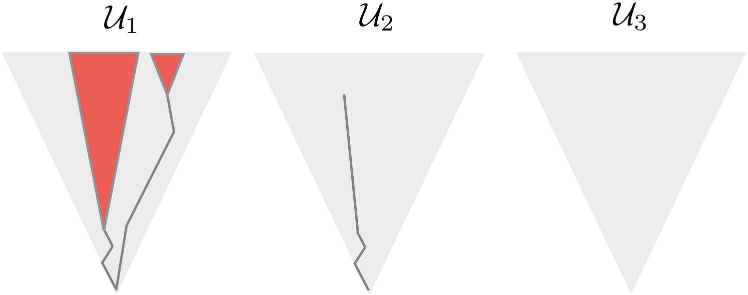
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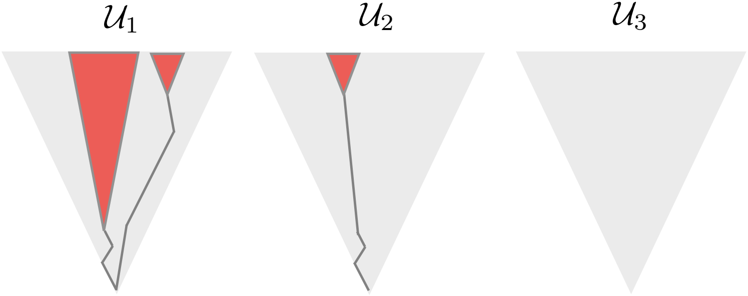
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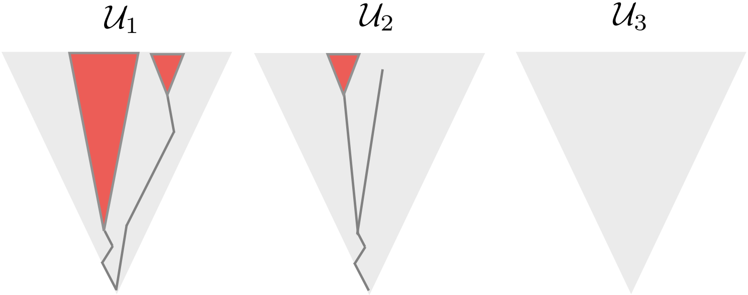
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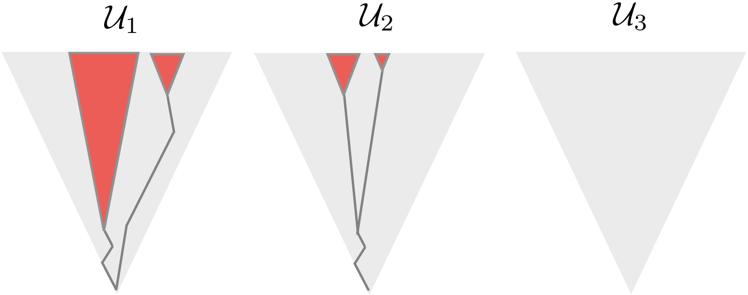
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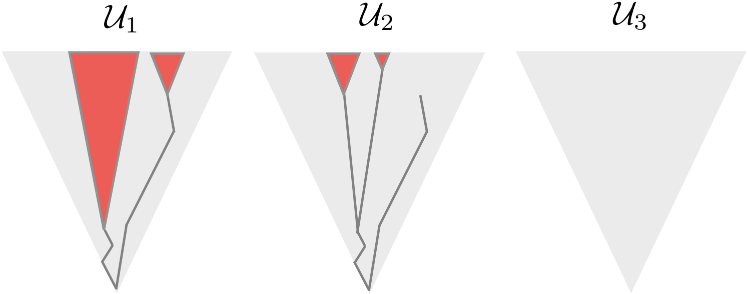

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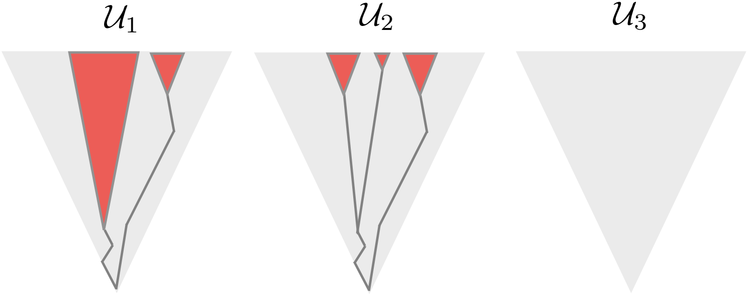
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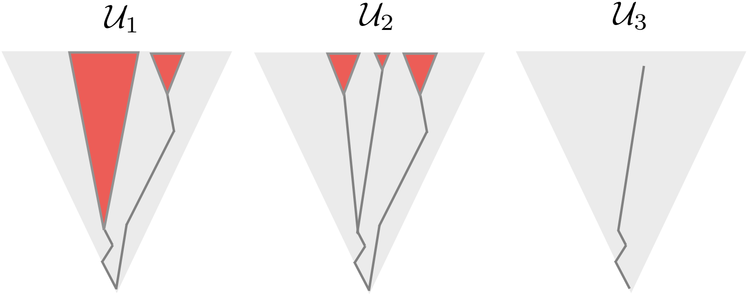

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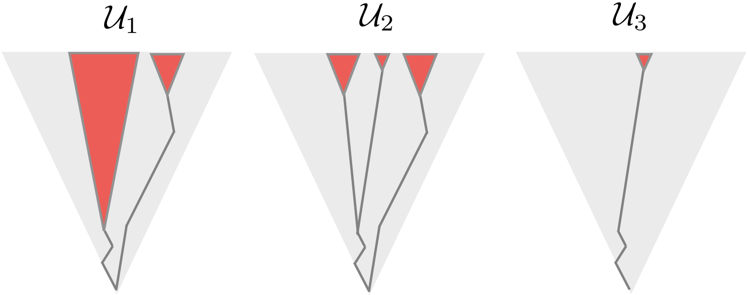
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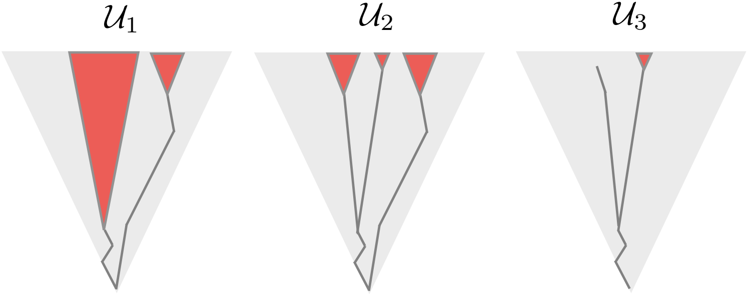
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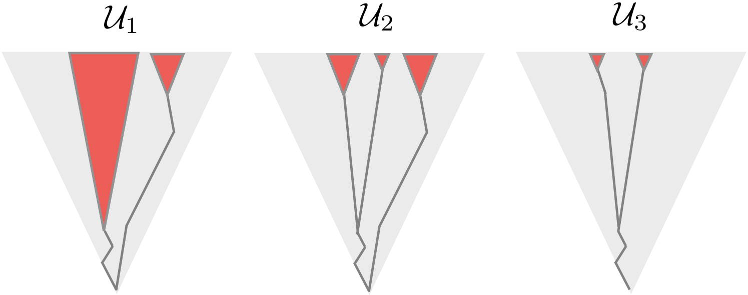
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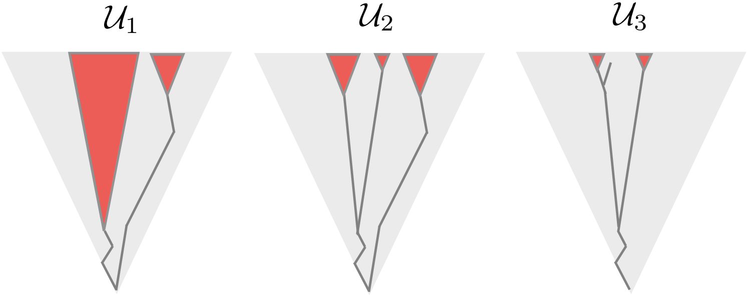
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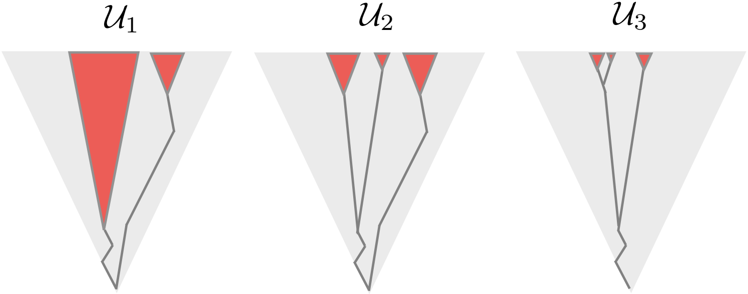
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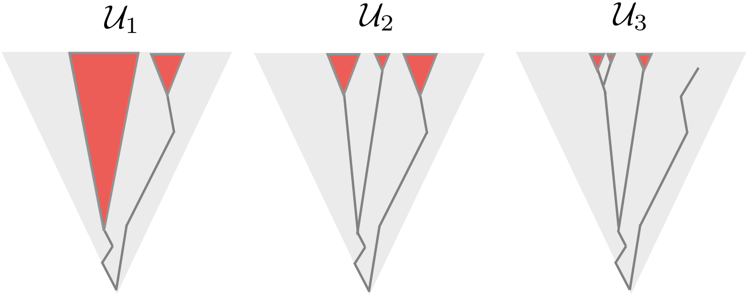
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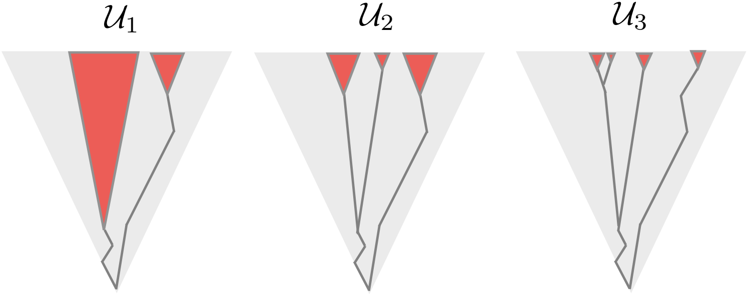
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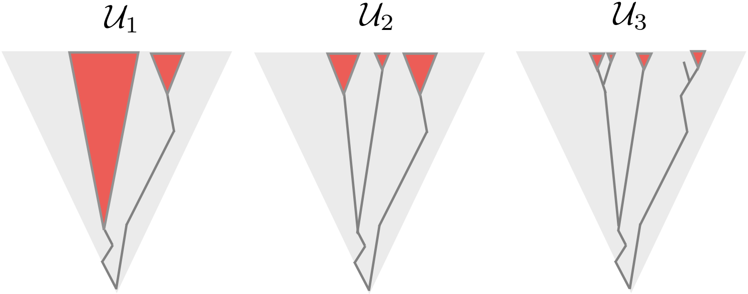
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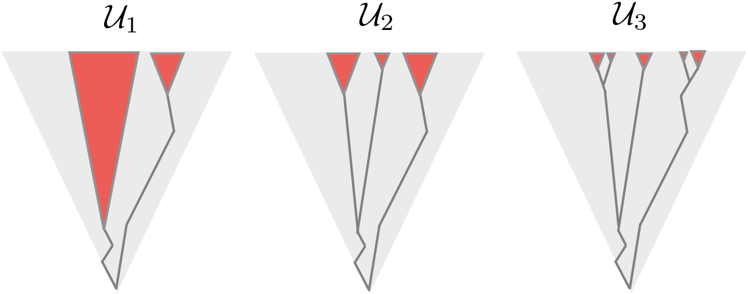

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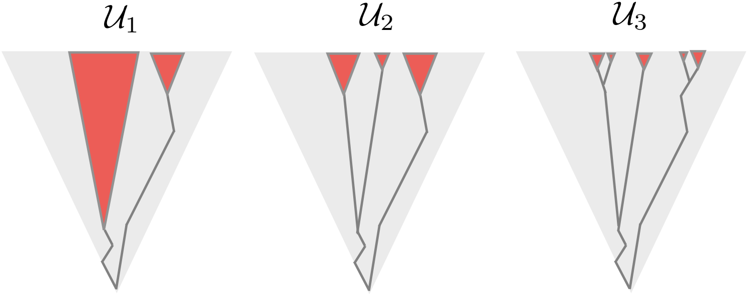

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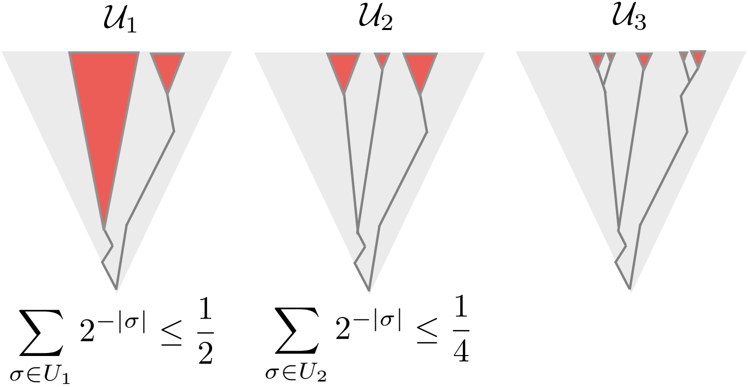
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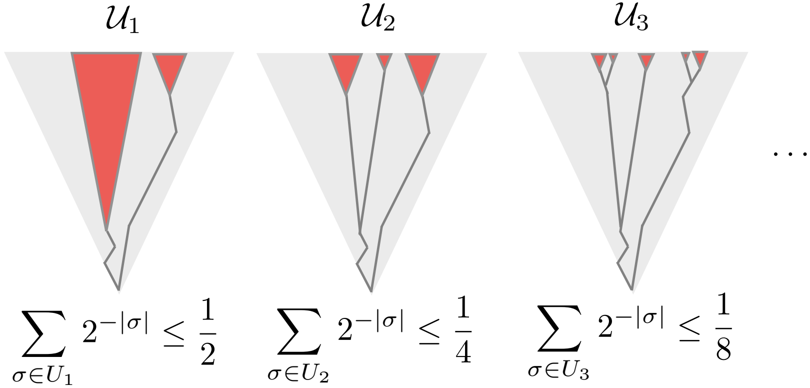
$\sum_{\sigma \in U_1} 2^{-|\sigma|} \leq \frac{1}{2}$
$\sum_{\sigma \in U_2} 2^{-|\sigma|} \leq \frac{1}{4}$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$\sum_{\sigma \in U_1} 2^{-|\sigma|} \leq \frac{1}{2}$
$\sum_{\sigma \in U_2} 2^{-|\sigma|} \leq \frac{1}{4}$
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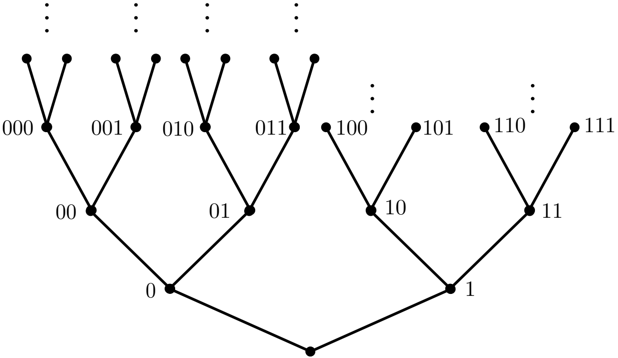
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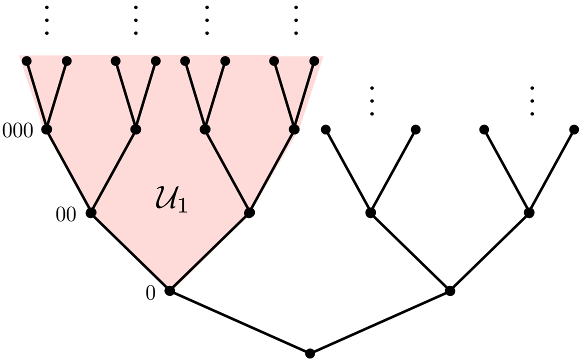
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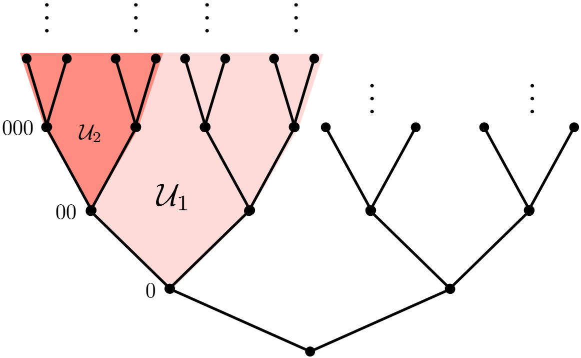

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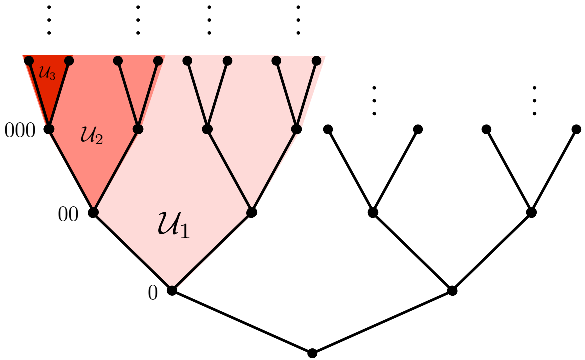

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Computable measures

We can also define Martin-Löf randomness with respect to any computable measure on 2^ω.

Definition

A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu([\![\sigma]\!])$ is computable as a real-valued function.

In other words, μ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2 = \{\frac{m}{2^n} : n, m \in \omega\}$ such that

$$|\mu([\![\sigma]\!]) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

From now on we will write $\mu(\sigma)$ instead of $\mu([\![\sigma]\!])$.

Randomness with respect to a computable measure

Definition

Let μ be a computable measure.

- A *μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes every μ-Martin-Löf test.

Two approaches to probabilistic computation

The standard definition of a probabilistic Turing machine is a non-deterministic Turing machine such that its transitions are chosen according to some probability distribution.

In the case of that this distribution is uniform, one can imagine that the machine is equipped with a fair coin that determines how it will transition from state to state.

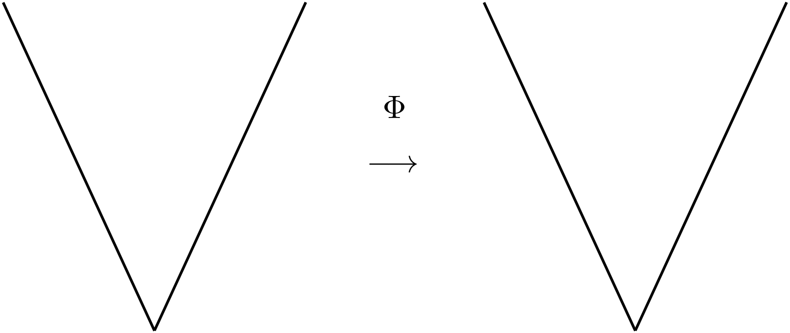
Alternatively, one can define a probabilistic machine to be an oracle Turing machine with some algorithmically random sequence as an oracle.

Key idea: For the purposes of computing a sequence or some sequence in a fixed collection *with positive probability*, these two approaches are equivalent.

Turing functionals

Definition

A *Turing functional* $\Phi : 2^\omega \to 2^\omega$ is given by a computably enumerable set S_Φ of pairs of strings (σ, τ) such that if $(\sigma, \tau), (\sigma', \tau') \in S_\Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$.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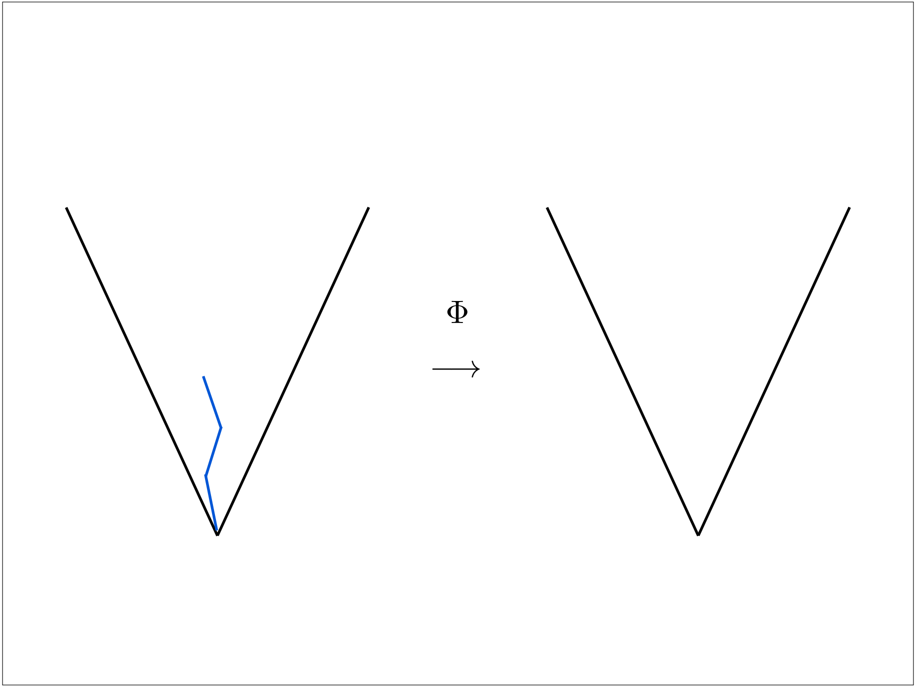
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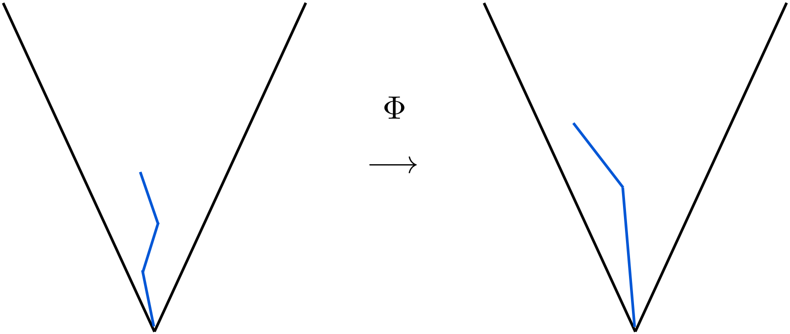
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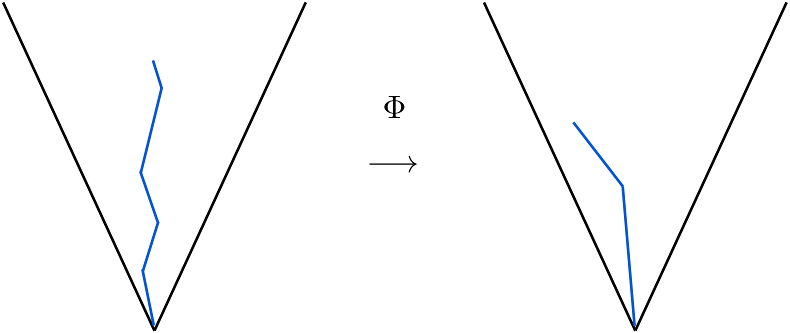
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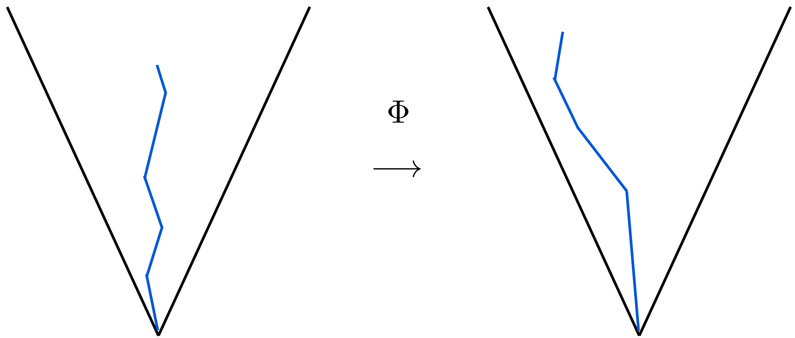
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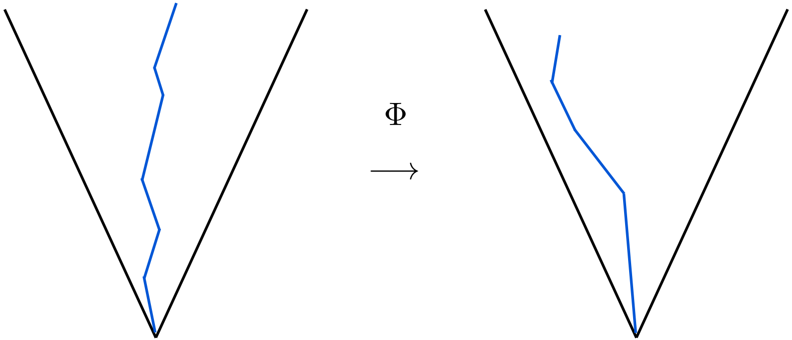
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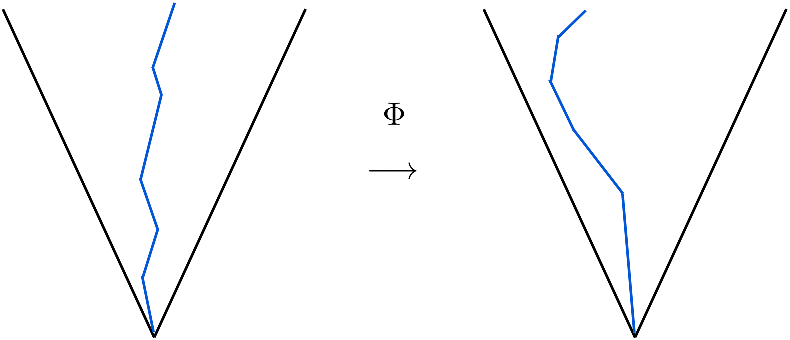
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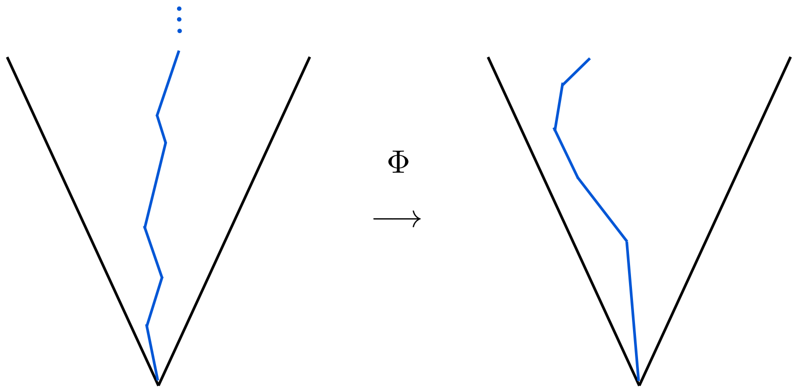
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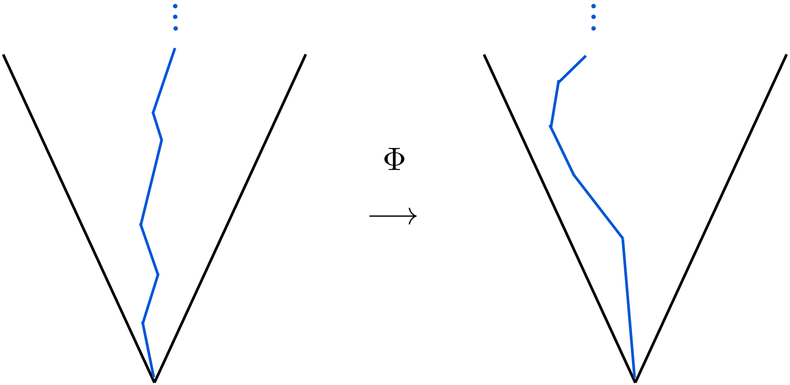
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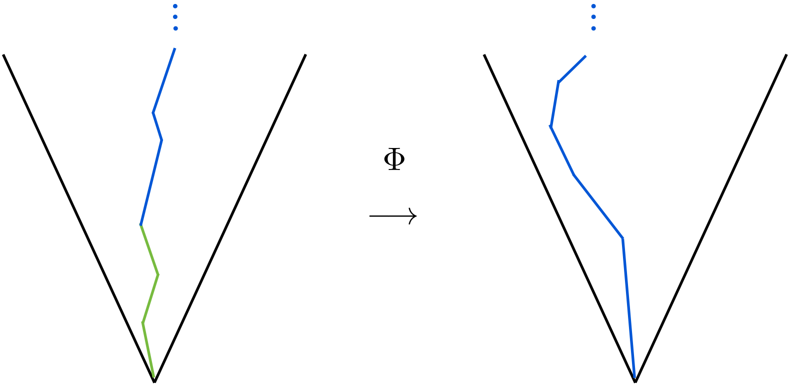
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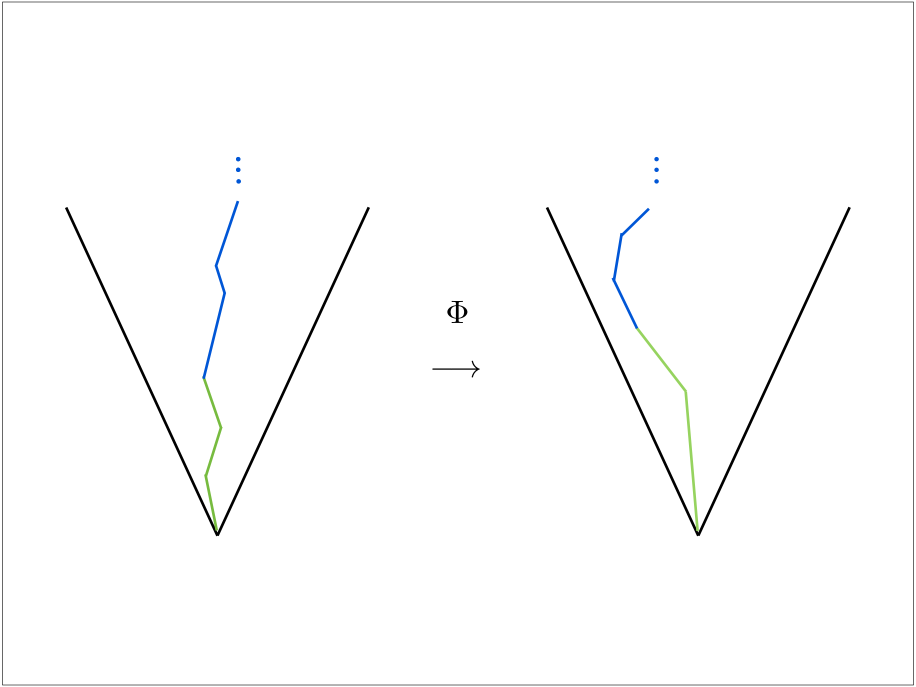
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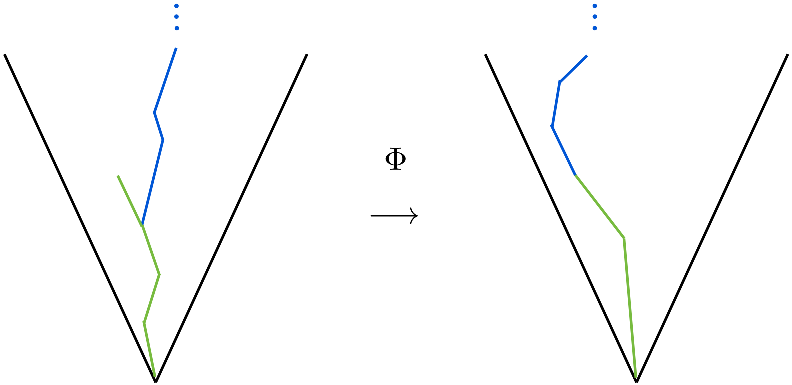
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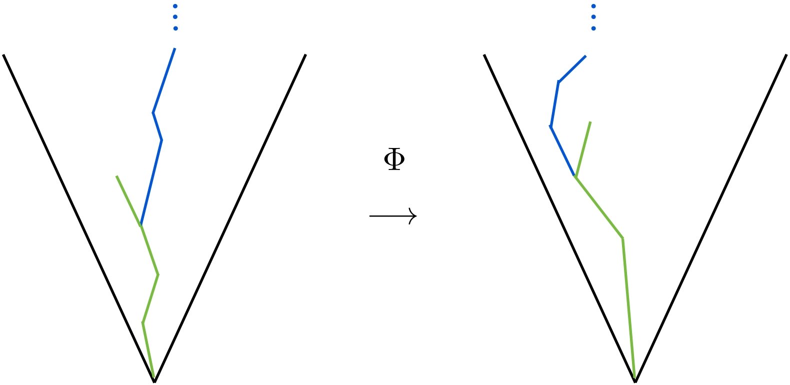
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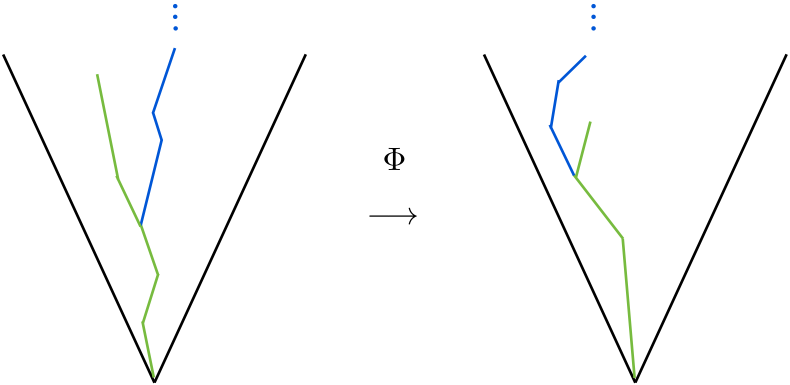
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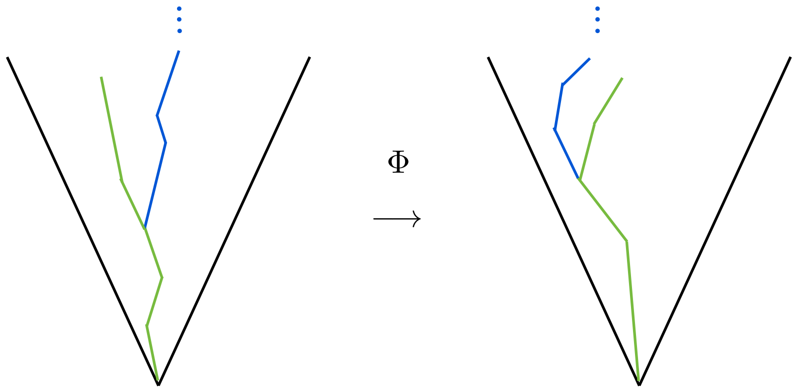
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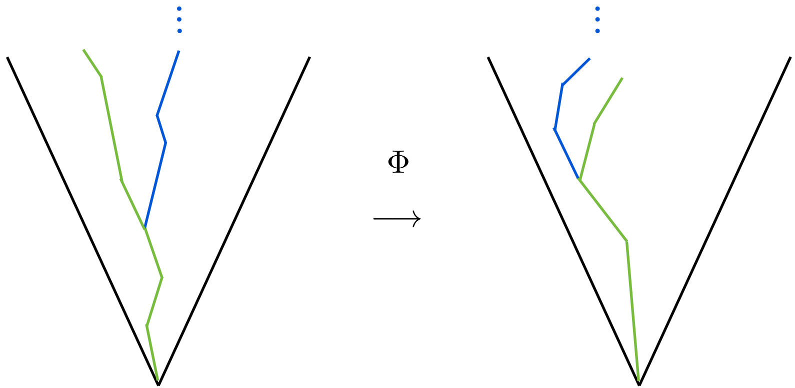
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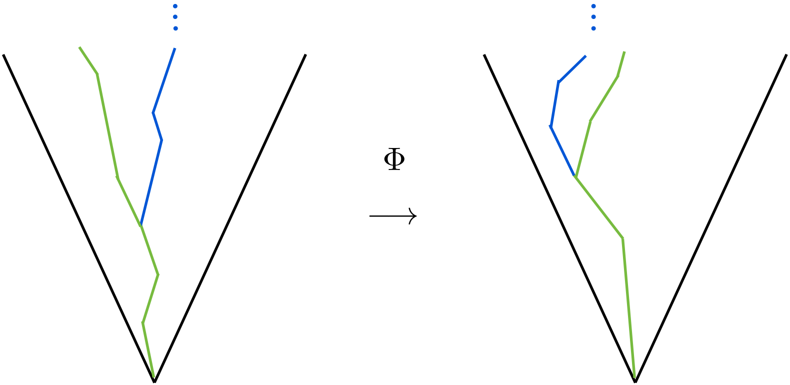
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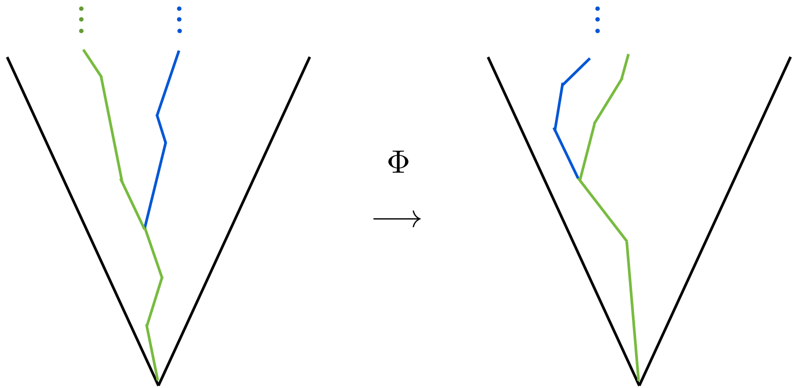
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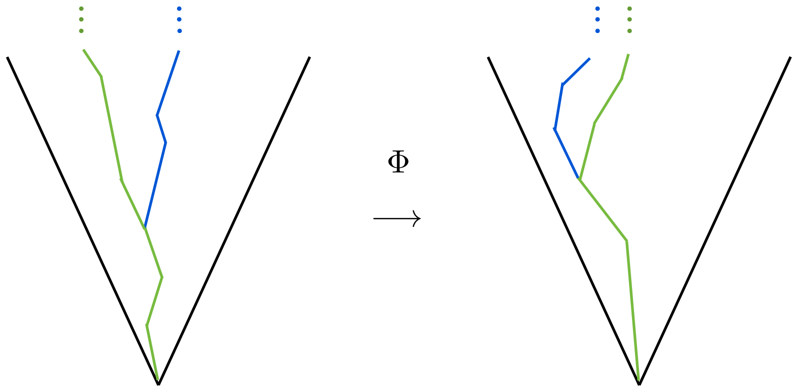
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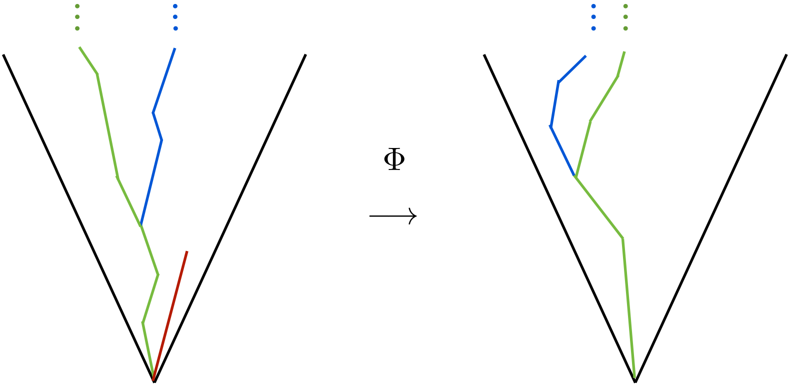
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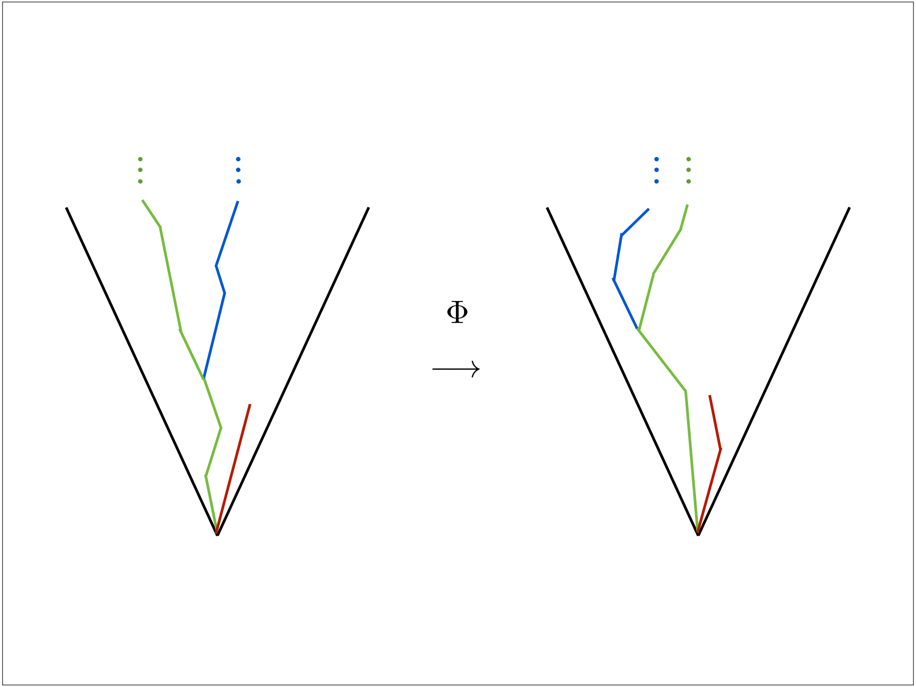
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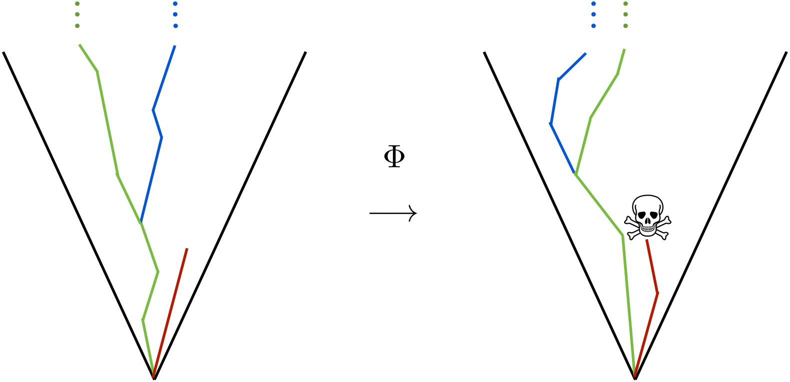
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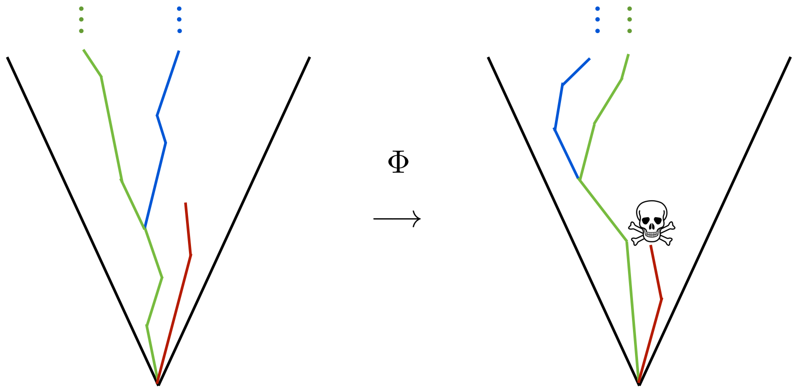
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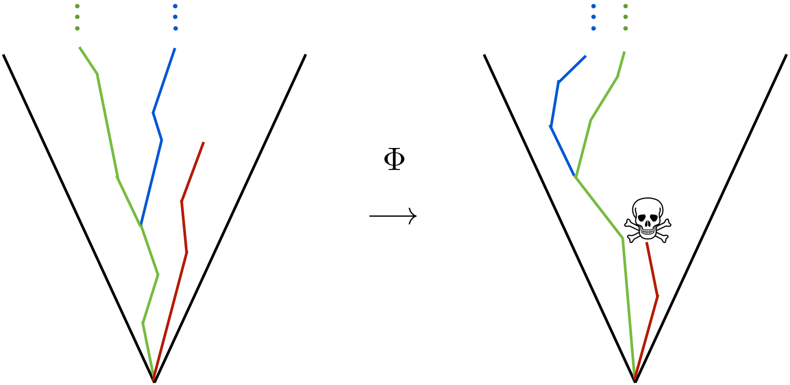
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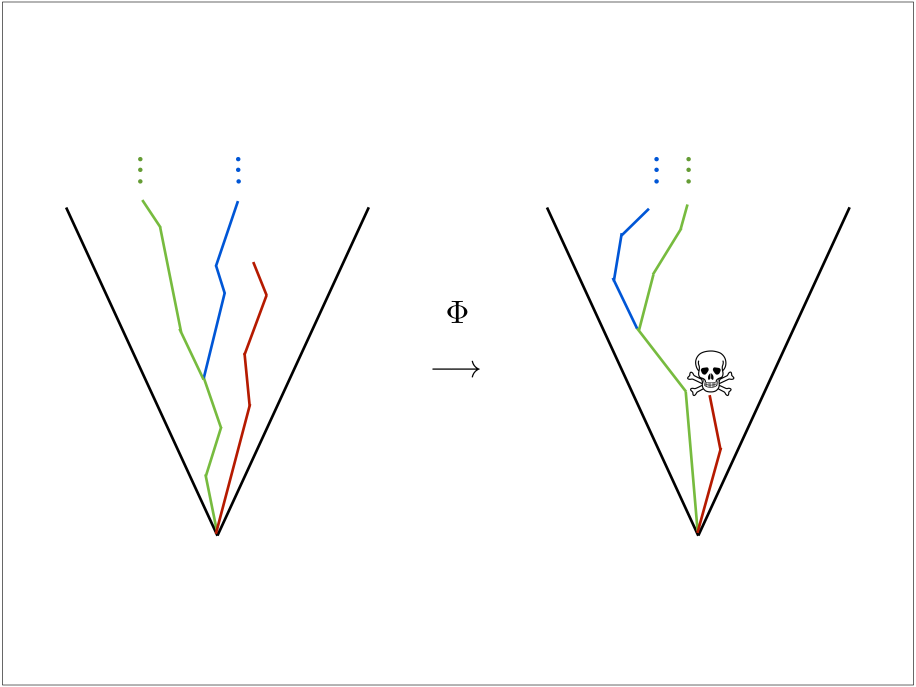
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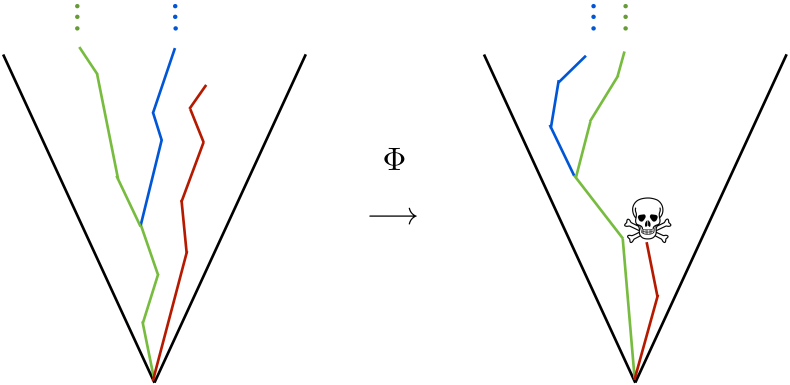
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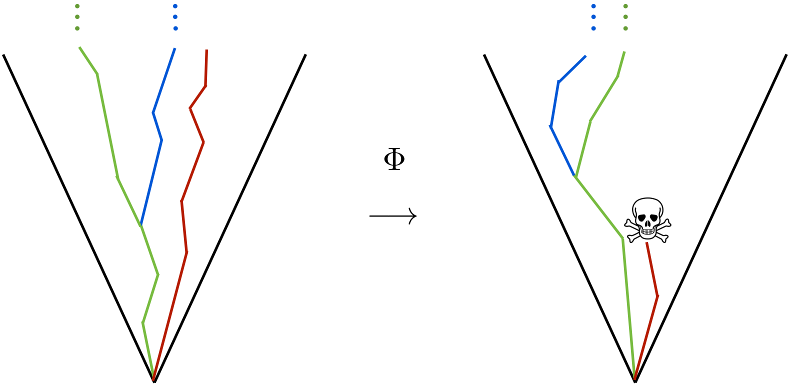
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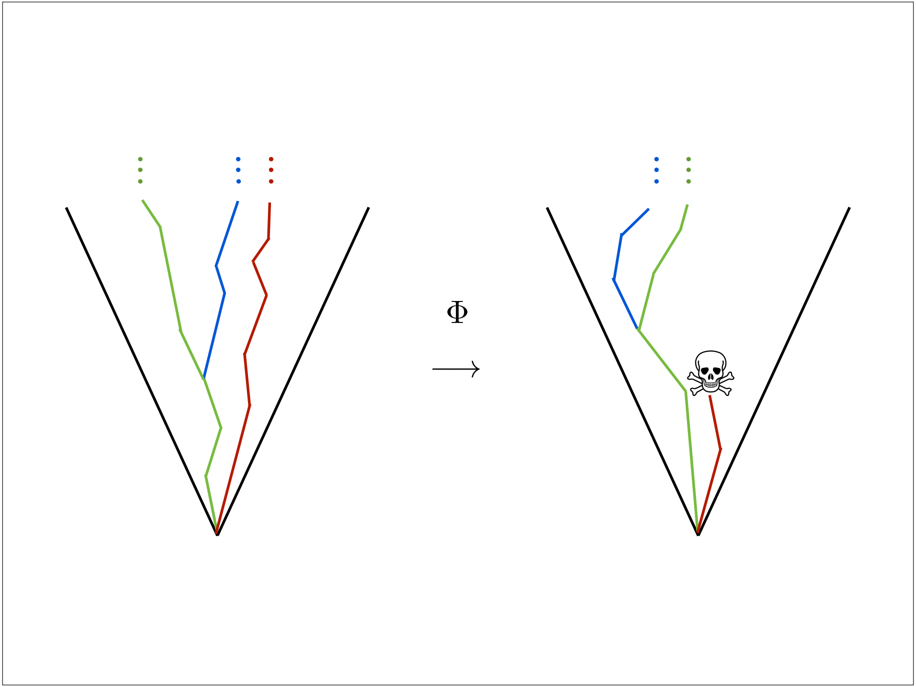
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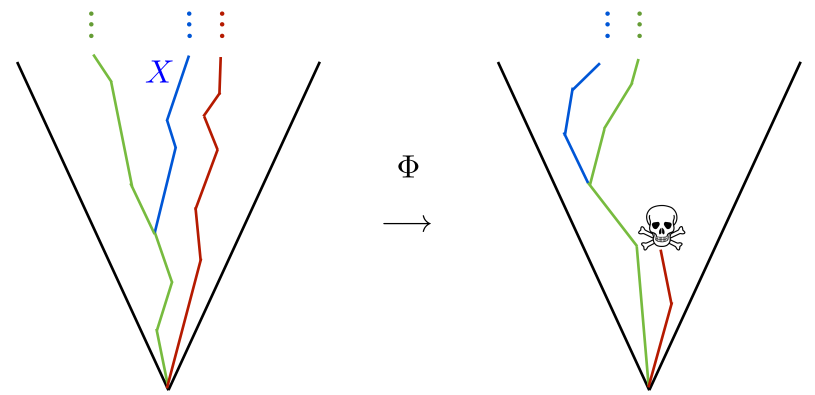
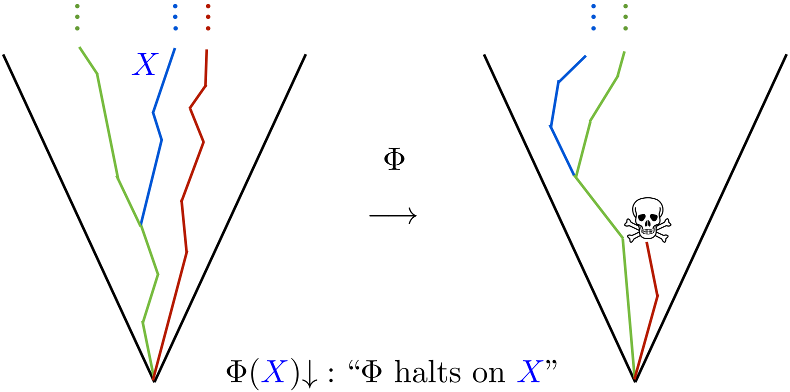
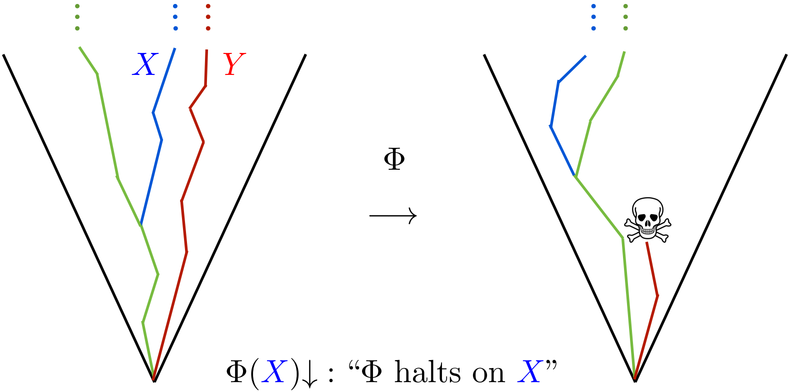
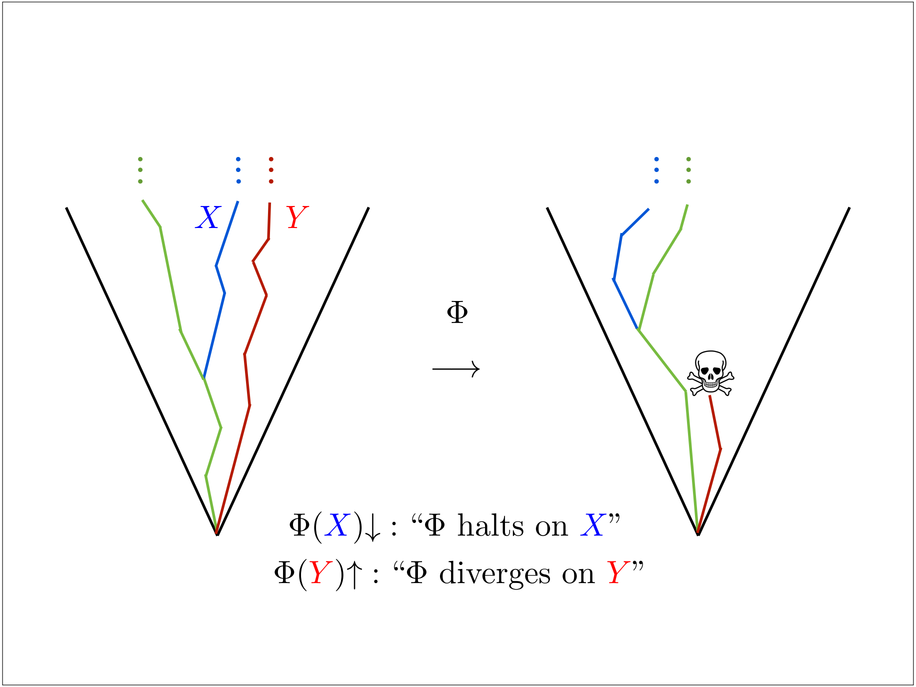


X
Φ
$\longrightarrow$


X
Φ
Φ(X)↓ : “Φ halts on X”


X
Y
Φ
Φ(X)↓ : “Φ halts on X”


X
Y
Φ
$\Phi(X)\downarrow$: "Φ halts on X"
$\Phi(Y)\uparrow$: "Φ diverges on Y"

Turing reducibility

If Φ is a Turing functional and $\Phi(B)\downarrow = A$, then we say that A is *Turing reducible* to B, denoted $A \leq_T B$.

"B computes A": $A \leq_T B$

One limitation: computing individual sequences

A sequence $A \in 2^\omega$ is *computable with positive probability* if

$$\lambda(\{X \in 2^\omega : A \leq_T X\}) > 0.$$

Theorem (Sacks)

A sequence is computable with positive probability if and only if it is computable.

Computing members of Π^0_1 classes

We cannot probabilistically compute any individual sequence that is not already computable.

However, the situation is more interesting when we consider the probabilistic computation of members of various Π^0_1 classes.

Π^0_1 classes

- $\mathcal{P} \subseteq 2^\omega$ is a Π^0_1 class if its complement is effectively open, i.e., the complement is given by a computable enumeration of basic open sets.
- Equivalently, $\mathcal{P} \subseteq 2^\omega$ is a Π^0_1 if it is the collection of infinite paths through through a computable tree (a subset of $2^{<\omega}$ that is closed downwards under $\preceq$).
- We can also define a Π^0_1 class to be the collection of infinite paths through a tree whose complement is computably enumerable.

Computationally powerful random sequences

It is worth noting that some Martin-Löf random sequences can compute a member of every Π^0_1 class.

- $X \in 2^\omega$ has *PA degree* if X computes a consistent completion of Peano arithmetic.
- Every sequence of PA degree computes a member of every Π^0_1 class.
- Some Martin-Löf random sequences have PA degree.

Dichotomy: A Martin-Löf random sequence has PA degree if and only if it computes the halting set $K = \{e : \phi_e(e){\downarrow}\}$.

However, by Sack's theorem, only measure zero many Martin-Löf random sequences have this property.

3. Negligible Π^0_1 classes

When probabilistic computation fails

$\mathcal{A} \subseteq 2^\omega$ is *negligible* if we cannot compute some member of $\mathcal{A}$ with positive probability.

That is,

$$\lambda\big(\{X \in 2^\omega : (\exists Y \in \mathcal{A})[Y \leq_T X]\}\big) = 0.$$

We can also provide a useful equivalent formulation of negligibility in terms of left-c.e. semi-measures.

Left-c.e. semi-measures

A *semi-measure* $\rho : 2^{<\omega} \to [0,1]$ satisfies

- $\rho(\Lambda) = 1$ and
- $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$ for every $\sigma \in 2^{<\omega}$.

A semi-measure ρ is *left-c.e.* if each value $\rho(\sigma)$ is the limit of a non-decreasing computable sequence of rationals, uniformly in σ.

Semi-measures and Turing functionals

For $\sigma \in 2^{<\omega}$, we define $\Phi^{-1}(\sigma) := \{X \in 2^{\omega} : \exists n\ (X{\upharpoonright}n, \sigma) \in S_{\Phi}\}$.

Proposition (Levin)

(i) *If Φ is a Turing functional, then λ_{Φ}, defined by*

$$\lambda_{\Phi}(\sigma) = \lambda(\Phi^{-1}(\sigma))$$

for every $\sigma \in 2^{<\omega}$, is a left-c.e. semi-measure.

(ii) *For every left c.e. semi-measure ρ, there is a Turing functional Φ such that $\rho = \lambda_{\Phi}$.*

A universal semi-measure

Levin proved the existence of a universal left-c.e. semi-measure.

A left-c.e. semi-measure M is *universal* if for every left-c.e. semi-measure ρ, there is some $c \in \omega$ such that

$$\rho(\sigma) \leq c \cdot M(\sigma)$$

for every $\sigma \in 2^{<\omega}$.

Defining negligibility in terms of semi-measures

Let M be a universal left-c.e. semi-measure.

Let $\overline{M}$ be the largest measure such that $\overline{M} \leq M$, which can be seen as a universal measure.

Proposition

$\mathcal{S} \subseteq 2^\omega$ is negligible if and only if $\overline{M}(\mathcal{S}) = 0$.

Members of negligible classes

A few observations:

- If a Π^0_1 class contains a computable member, it cannot be negligible.
- Moreover, if a Π^0_1 class contains a Martin-Löf random member, it cannot be negligible, since any Π^0_1 class with a random member must have positive Lebesgue measure.

These two facts are subsumed by the following result:

Proposition (Bienvenu, Porter)

Let $\mathcal{P}$ be a negligible Π^0_1 class. Then for every computable measure μ, $\mathcal{P}$ contains no $X \in \mathsf{MLR}_\mu$.

Does the converse hold?

Suppose that $\mathcal{P}$ is a Π^0_1 class such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure μ.

Does it follow that $\mathcal{P}$ is negligible? No.

Theorem (Bienvenu, Porter)

There is a non-negligible Π^0_1 class $\mathcal{P}$ such that $\mathcal{P} \cap \mathsf{MLR}_\mu = \emptyset$ for every computable measure μ.

Computing members of negligible Π^0_1 classes

As mentioned above, some Martin-Löf random sequences can compute a member of every Π^0_1 class (but only measure zero many random sequences have this property).

If we consider a slightly stronger notion of randomness known as *weak 2-randomness*, we get a stronger result.

Weak 2-randomness is the notion of randomness that results from replacing Martin-Löf tests with *generalized Martin-Löf tests*: a collection $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that $\lambda(\mathcal{U}_i) \to 0$.

Theorem (Bienvenu, Porter)

If $X \in 2^\omega$ is weakly 2-random, then X cannot compute any member of a negligible Π^0_1 class.

3. Deep Π^0_1 classes

Deep classes: the idea

Unlike negligibility, we only define depth for Π^0_1 classes.

Depth is a property that is strictly stronger than negligibility for Π^0_1 classes.

Instead of considering how difficult it is to produce a path through a Π^0_1 class $\mathcal{P}$, we can consider how difficult it is to produce an *initial segment* of some path through $\mathcal{P}$, level by level.

Deep Π^0_1 classes are the "most difficult" Π^0_1 classes in this respect.

Some notation

Let $\mathcal{P} \subseteq 2^\omega$ be a Π^0_1 class.

Let $T_\mathcal{P} \subseteq 2^{<\omega}$ be the set of extendible nodes of $\mathcal{P}$,

$$T_\mathcal{P} = \{\sigma \in 2^{<\omega} : [\![\sigma]\!] \cap \mathcal{P} \neq \emptyset\}.$$

Thus $T_\mathcal{P}$ is the canonical co-c.e. tree such that $\mathcal{P} = [T_\mathcal{P}]$ (the set of infinite paths through $T_\mathcal{P}$).

Hereafter T will stand for $T_\mathcal{P}$.

For each $n \in \omega$, T_n consists of all strings in T of length n.

Deep classes: the definition

Let $\mathcal{P}$ be a Π^0_1 class and let T be the canonical co-c.e. tree such that $\mathcal{P} = [T]$.

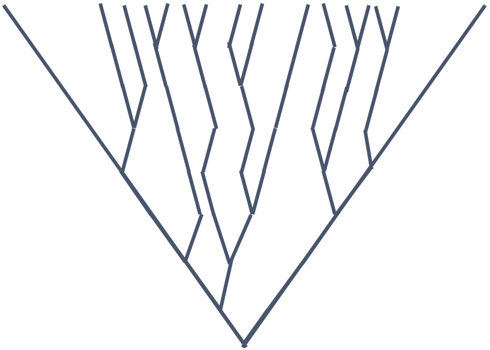
$\mathcal{P}$ is a *deep class* if there is some computable order function $h : \omega \to \omega$ (that is, a computable non-decreasing, unbounded function) such that

$$M(T_n) \leq 2^{-h(n)},$$

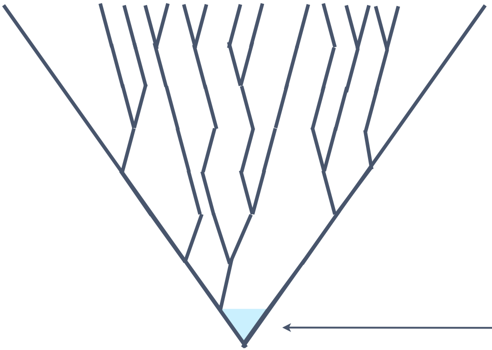
where $M(T_n) = \sum_{\sigma \in T_n} M(\sigma)$.

That is, the probability of producing some initial segment of a path through $\mathcal{P}$ is effectively bounded from above.

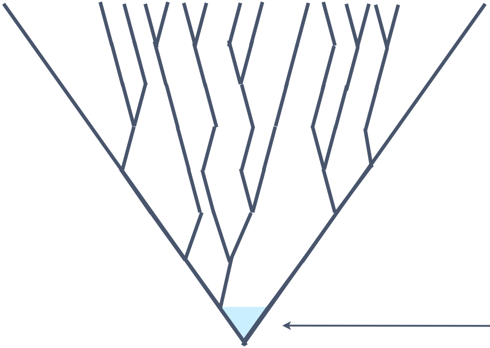
Note: Every deep class is negligible.



Length Probability

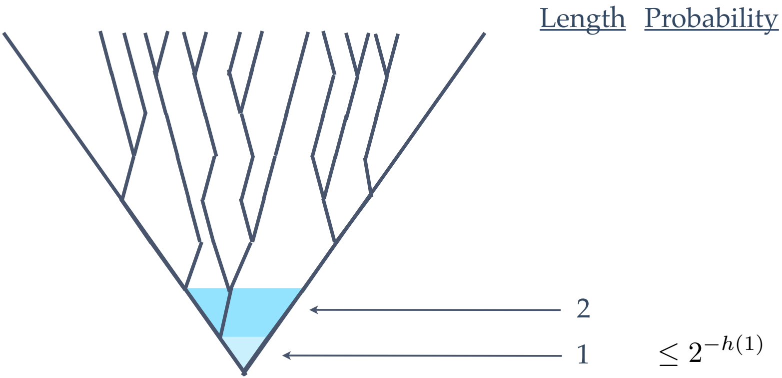


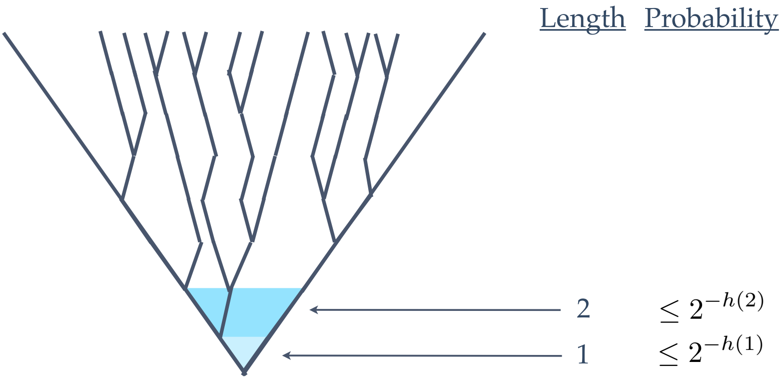
Length	Probability
1	



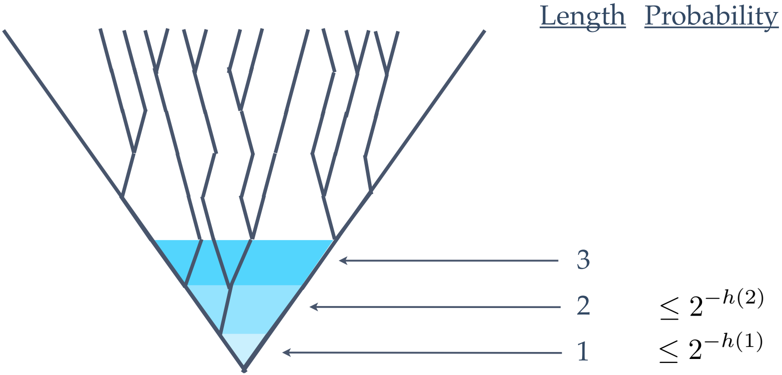
Length Probability

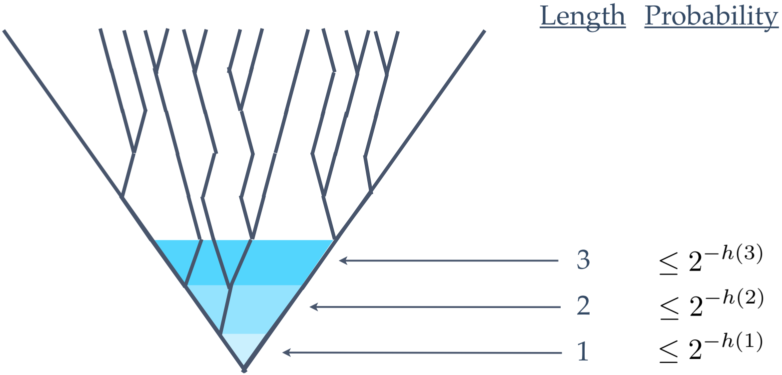
1 $\leq 2^{-h(1)}$

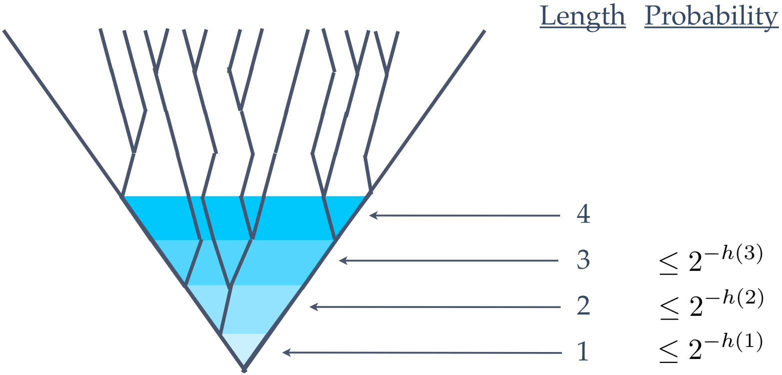

Length
Probability
2
1
$\leq 2^{-h(1)}$

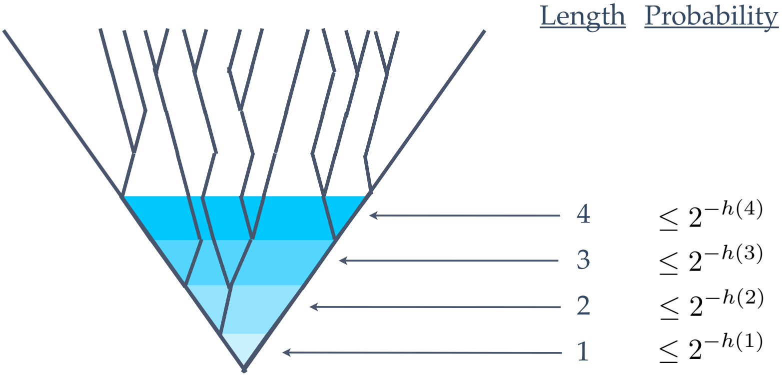


Length
Probability
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$

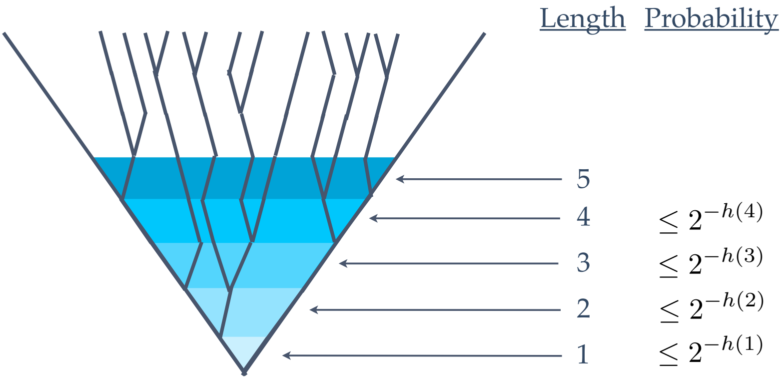

Length
Probability
3
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$


Length
Probability
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$

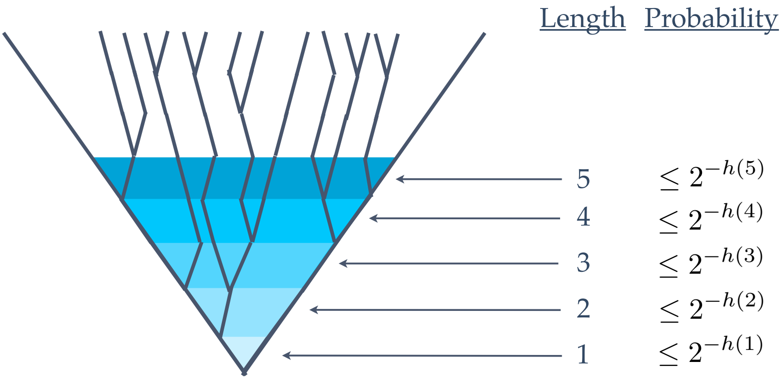

Length
Probability
4
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$



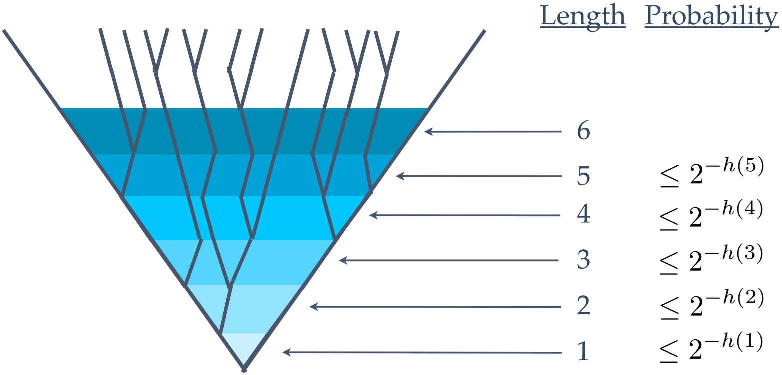
Length
Probability
4
$\leq 2^{-h(4)}$
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$



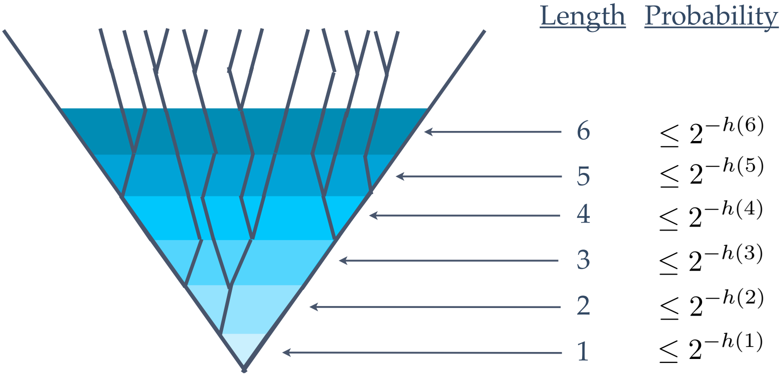
Length
Probability
5
4
$\leq 2^{-h(4)}$
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$

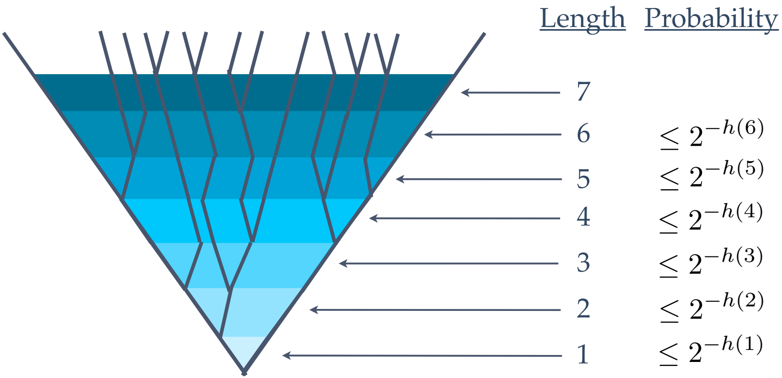


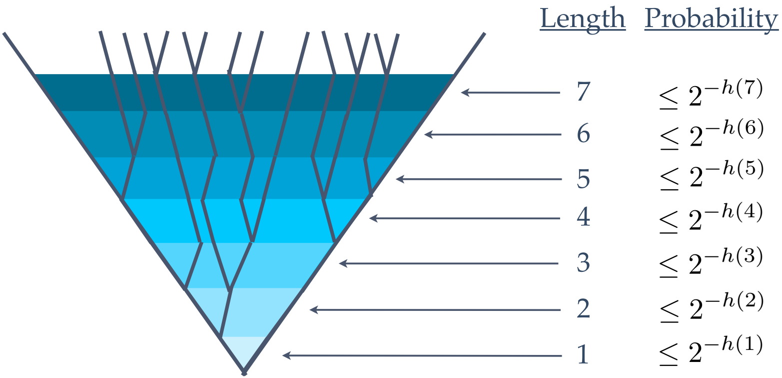
Length
Probability
5
$\leq 2^{-h(5)}$
4
$\leq 2^{-h(4)}$
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$



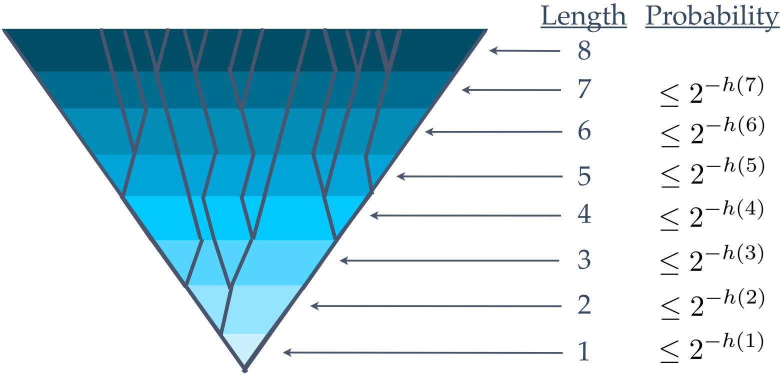
Length
Probability
6
5
4
3
2
1
$\leq 2^{-h(5)}$
$\leq 2^{-h(4)}$
$\leq 2^{-h(3)}$
$\leq 2^{-h(2)}$
$\leq 2^{-h(1)}$

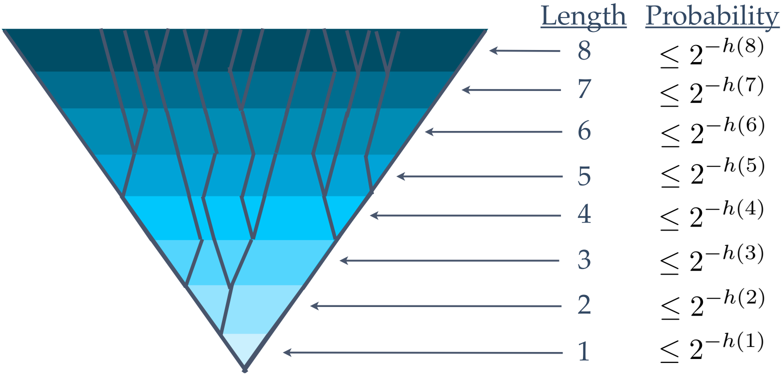

Length
Probability
6
$\leq 2^{-h(6)}$
5
$\leq 2^{-h(5)}$
4
$\leq 2^{-h(4)}$
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$


Length
Probability
7
6
$\leq 2^{-h(6)}$
5
$\leq 2^{-h(5)}$
4
$\leq 2^{-h(4)}$
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$



Length
Probability
7
$\leq 2^{-h(7)}$
6
$\leq 2^{-h(6)}$
5
$\leq 2^{-h(5)}$
4
$\leq 2^{-h(4)}$
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$


Length
Probability
8
7
$\leq 2^{-h(7)}$
6
$\leq 2^{-h(6)}$
5
$\leq 2^{-h(5)}$
4
$\leq 2^{-h(4)}$
3
$\leq 2^{-h(3)}$
2
$\leq 2^{-h(2)}$
1
$\leq 2^{-h(1)}$



Length Probability
8 $\leq 2^{-h(8)}$
7 $\leq 2^{-h(7)}$
6 $\leq 2^{-h(6)}$
5 $\leq 2^{-h(5)}$
4 $\leq 2^{-h(4)}$
3 $\leq 2^{-h(3)}$
2 $\leq 2^{-h(2)}$
1 $\leq 2^{-h(1)}$

Why use the co-c.e. tree in the definition of depth?

For every Π^0_1 class $\mathcal{P}$ there is a computable tree $S \subseteq 2^{<\omega}$ such that $\mathcal{P} = [S]$.

Why can't we use this computable tree S in the definition of depth?

First, in general, S will contain non-extendible nodes, so even if we can compute some element in S_n, we still may fail to compute an initial segment of a member of $\mathcal{P}$.

But this observation doesn't rule out the possibility that we can define depth in terms of computable trees.

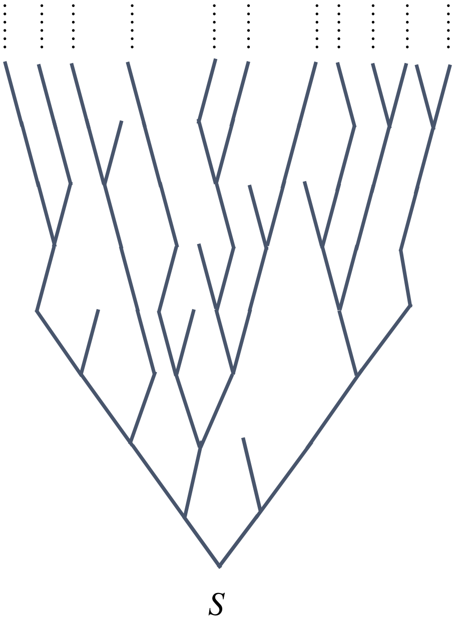
A better reason

Theorem (Bienvenu, Porter)

Let S be a computable tree. Then there is no computable order h such that $M(S_n) \leq 2^{-h(n)}$ for every $n \in \omega$.

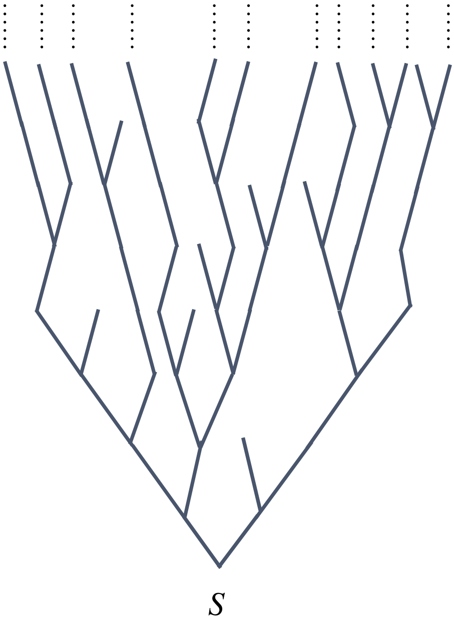
Corollary (Bienvenu, Porter)

Let S be a tree with a computable sub-tree. Then there is no computable order function h such that $M(S_n) \leq 2^{-h(n)}$ for every $n \in \omega$.



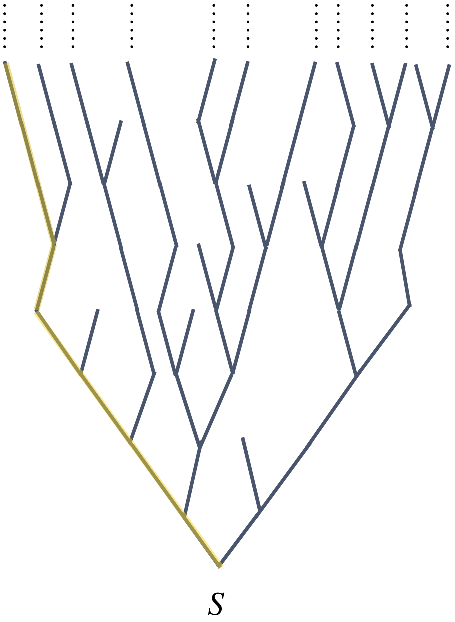
S

Case 1: S has only finitely many non-extendible nodes.

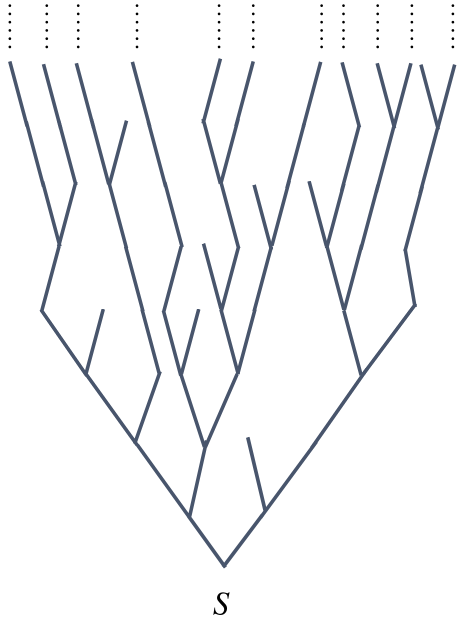


Case 1: S has only finitely many non-extendible nodes.

In this case, the left-most path of S is computable.

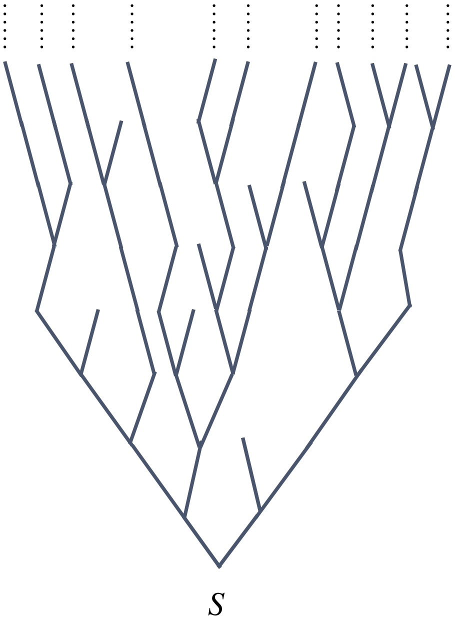


Case 2: S has infinitely many non-extendible nodes.



Case 2: S has infinitely many non-extendible nodes.

In this case, first we find a sequence of non-extendible nodes of increasing length.



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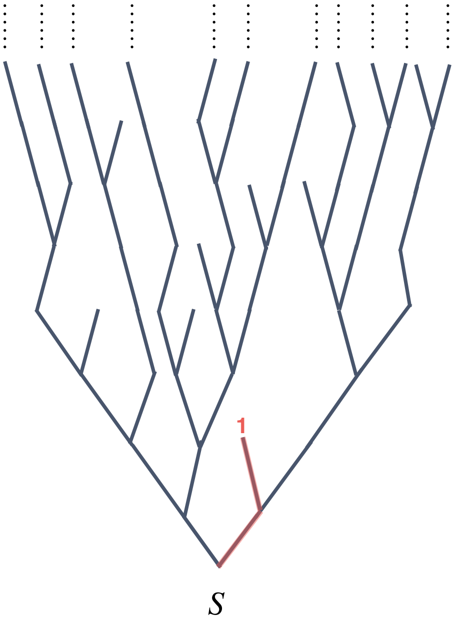

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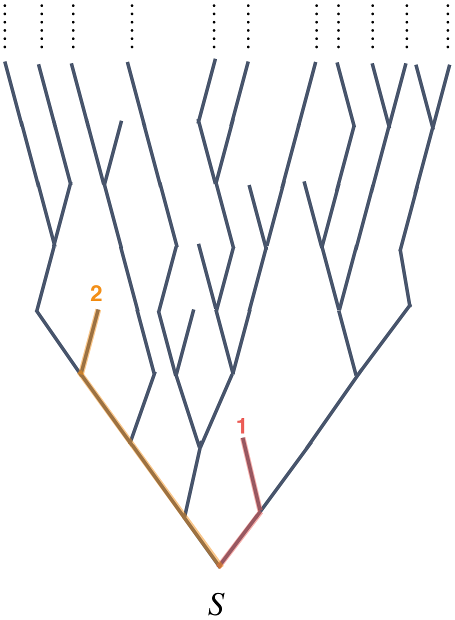

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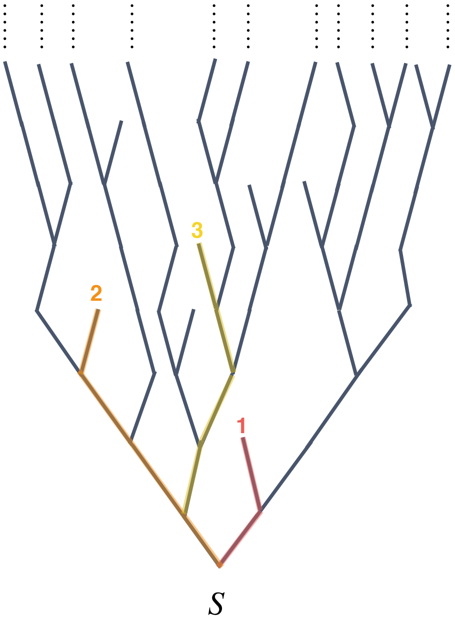

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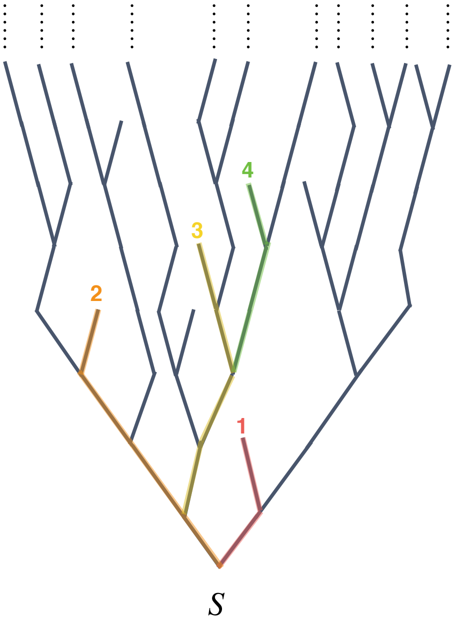

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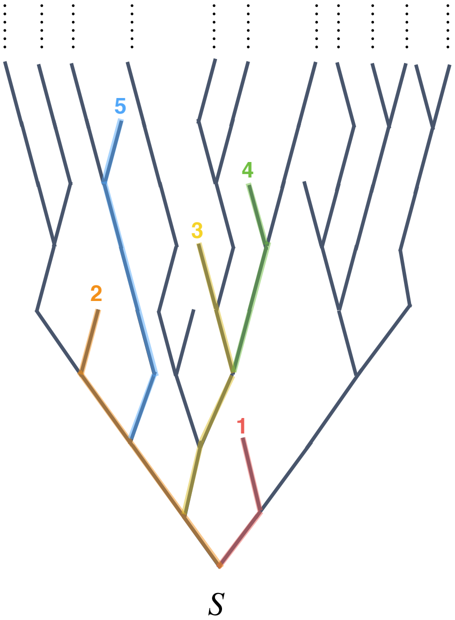

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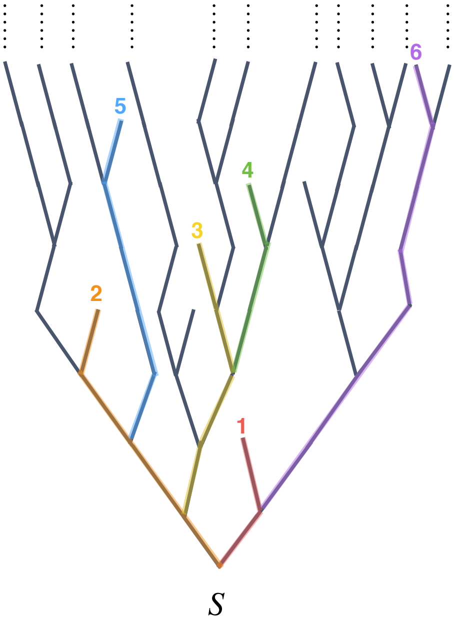
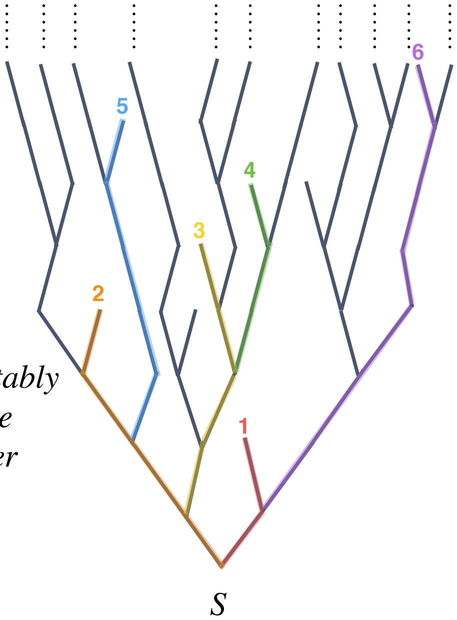


Case 2: S has infinitely many non-extendible nodes.

In this case, first we find a sequence of non-extendible nodes of increasing length.

Next let $f : \mathbb{N} \to \mathbb{N}$ *be a function that is computably approximable from above with no computable lower bound such that*

$$\sum_{n\in\mathbb{N}} 2^{-f(n)} \leq 1.$$



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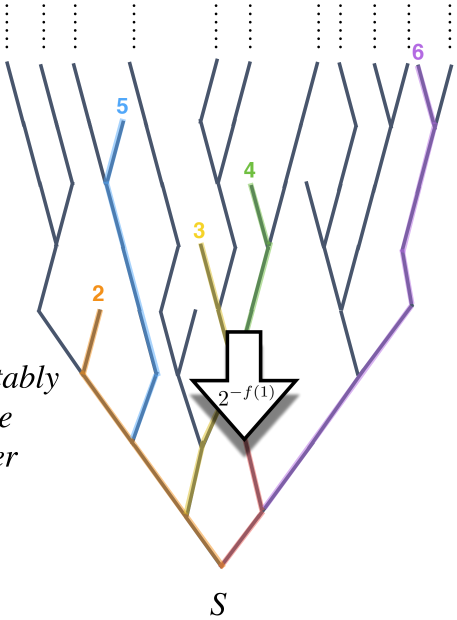

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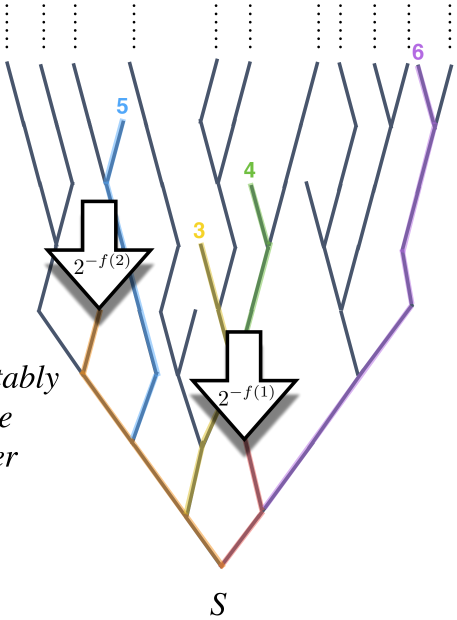

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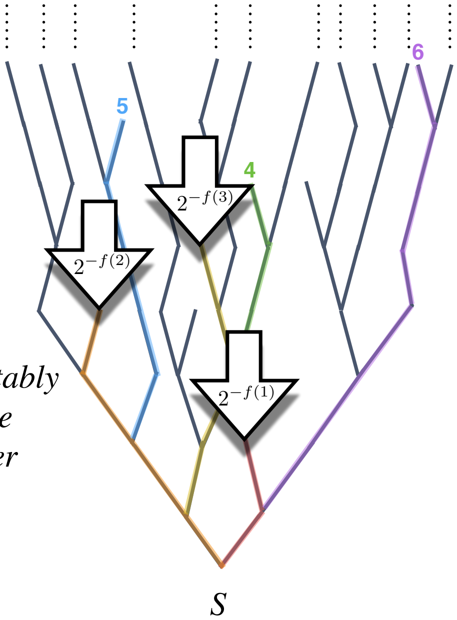

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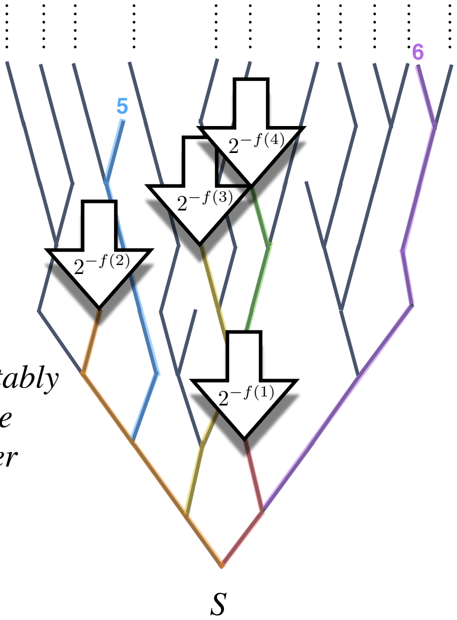
$$\sum_{n \in \mathbb{N}} 2^{-f(n)} \leq 1.$$

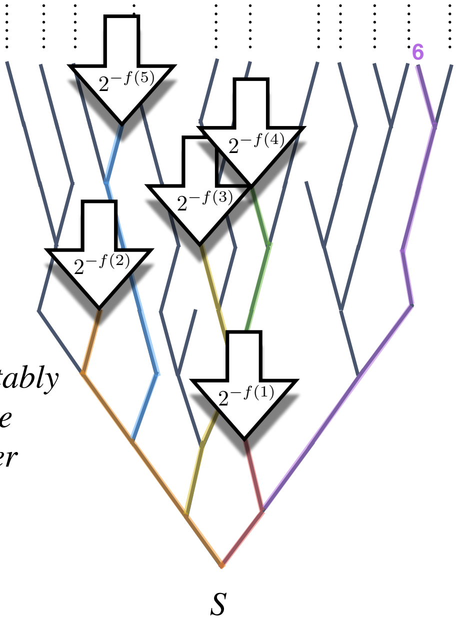


Case 2: S has infinitely many non-extendible nodes.

In this case, first we find a sequence of non-extendible nodes of increasing length.

Next let $f : \mathbb{N} \to \mathbb{N}$ *be a function that is computably approximable from above with no computable lower bound such that*

$$\sum_{n \in \mathbb{N}} 2^{-f(n)} \leq 1.$$



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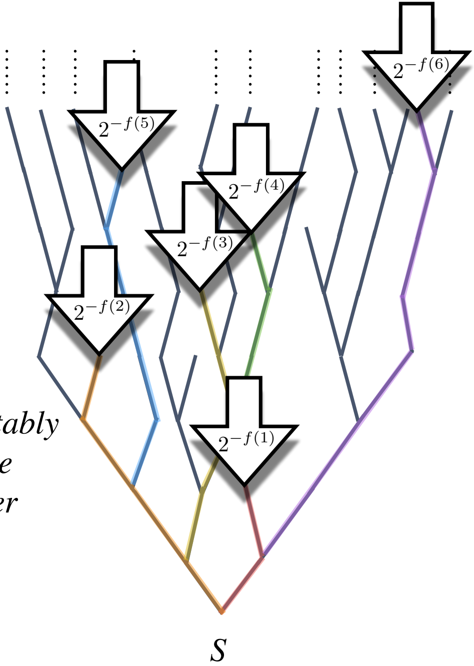


Does such an f exist?

Yes!

Let U be a universal, prefix-free Turing machine.

For each $\sigma \in 2^{<\omega}$, the *prefix-free Kolmogorov complexity* of σ is defined to be

$$K(\sigma) := \min\{|\tau| : U(\tau) = \sigma\}.$$

If $(\sigma_i)_{i \in \omega}$ is an enumeration of $2^{<\omega}$ in length-lexicographical order, then

$$f(i) = K(\sigma_i)$$

is the desired function f.

Depth vs. negligibility

It is not hard to show that every deep class is negligible.

Is every negligible class deep? No.

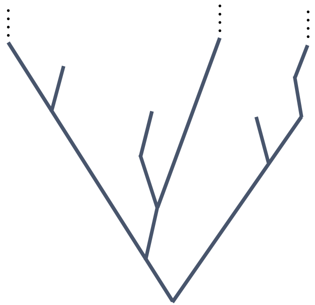
Theorem (Bienvenu, Porter)

For every deep class $\mathcal{P}$, there is negligible class $\mathcal{Q}$ that is not deep such that

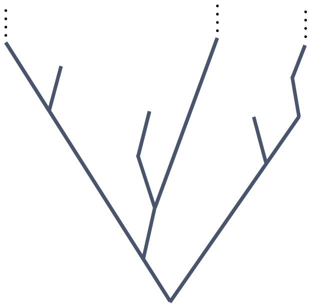
- *for every $X \in \mathcal{P}$, we have $X \in \mathcal{Q}$, and*
- *for every $Y \in \mathcal{Q}$, $Y = \sigma^\frown X$ for some $\sigma \in 2^{<\omega}$ and $X \in \mathcal{P}$.*

In other words, every deep class is Muchnik equivalent to a negligible Π^0_1 class that is not deep.

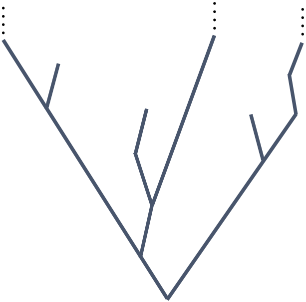
However, it is worth noting that depth is preserved under Medvedev equivalence.



S

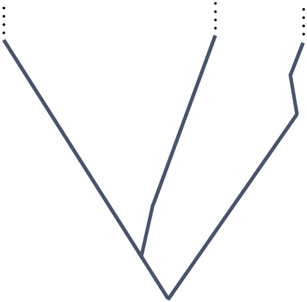


S
computable
tree

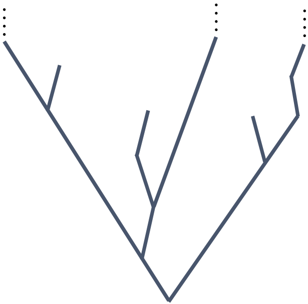


S

computable
tree

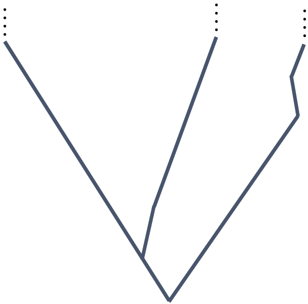


T



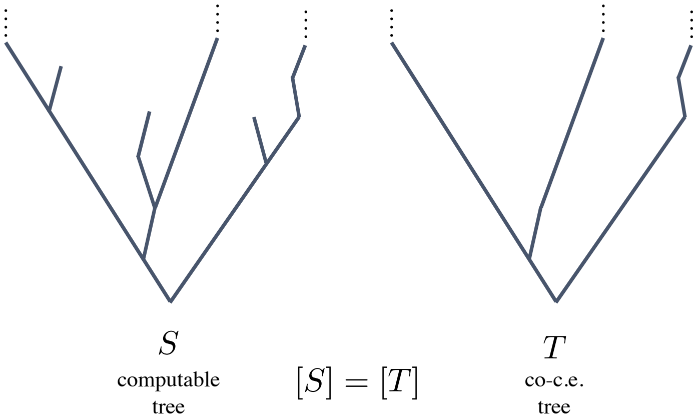
S

computable tree

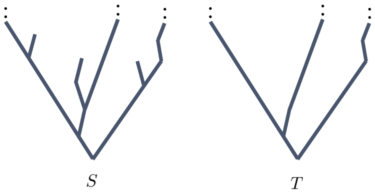


T

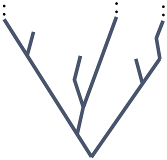
co-c.e. tree



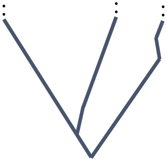
S
computable tree
$[S] = [T]$
T
co-c.e. tree



S
T



S



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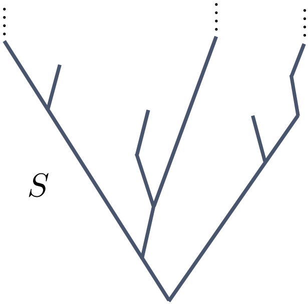
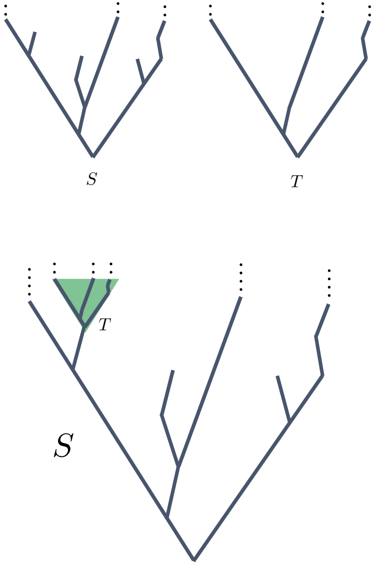
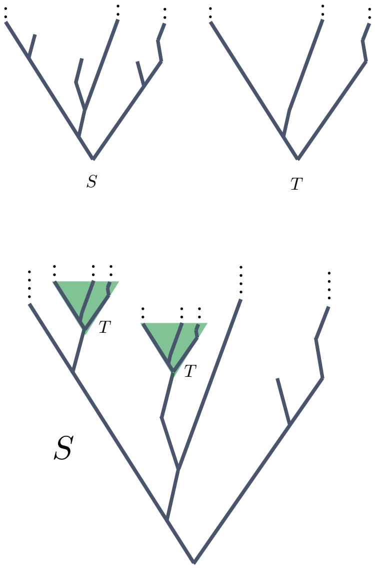
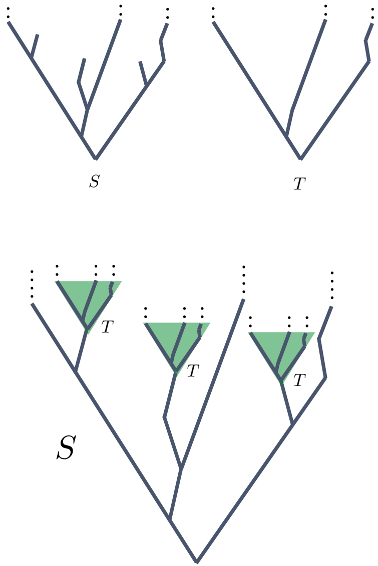


S


S
T
T
S



S
T
T
T
S


S
T
T
T
T
S

Computing members of deep Π^0_1 classes

What level of randomness $\mathcal{R}$ guarantees that no $\mathcal{R}$-random sequence can compute a member of a deep Π^0_1 class?

The answer is known as *difference randomness*, which is formulated in terms of *difference tests*: a collection of pairs of uniformly effectively open subsets $(\mathcal{U}_i, \mathcal{V}_i)_{i\in\omega}$ of 2^ω such that $\lambda(\mathcal{U}_i \setminus \mathcal{V}_i) \leq 2^{-i}$.

Theorem (Bienvenu, Porter)

If $X \in 2^\omega$ is difference random, then X cannot compute any member of a deep Π^0_1 class.

Note: The difference random sequences are precisely the Martin-Löf random sequences that cannot compute a completion of PA.

Paradigm example of a deep class: Consistent completions of PA

The following is implicit in work of Levin and Stephan.

Theorem

The Π^0_1 class of consistent completions of PA is a deep class.

What exactly does this tell us?

Not only can we not probabilistically compute some consistent completion of *PA* with positive probability, but we cannot even hope to produce longer and longer initial segments of a consistent completion of *PA* with sufficiently high probability.

Shift-complex sequences: the idea

A Martin-Löf random sequence X has high initial segment complexity, satisfying

$$K(X \upharpoonright n) \geq n - O(1).$$

Nonetheless, X will still contain arbitrarily long runs of 0s (since all Martin-Löf random sequences are normal).

That is, certain subwords of X can have fairly low initial segment complexity.

By contrast, a shift-complex sequence is a sequence with the property that every subword has high initial segment complexity.

Shift-complex sequences: the formal definition

For $\delta \in (0,1)$ and $c \in \omega$, we say that $X \in 2^\omega$ is *(δ, c)-shift complex* if

$$K(\tau) \geq \delta|\tau| - c$$

for every subword τ of X.

The following draws upon work of Rumyantsev.

Theorem (Bienvenu, Porter)

For every $\delta \in (0,1)$ and $c \in \omega$, the (δ, c)-shift complex sequences form a deep class.

Diagonally non-computable sequences and randomness

Recall that a sequence X is diagonally non-computable if there is some total function $f \leq_T X$ such that $f(e) \neq \phi_e(e)$ for every e.

Every Martin-Löf random sequence X is diagonally non-computable:

Let $f(e) = X{\upharpoonright}e$ (coded as a natural number).

Note that $f(e) < 2^{e+1}$.

DNC_h functions

Let h be a computable, non-decreasing, unbounded function.

f is a DNC_h function if

- f is total,
- $f(e) \neq \phi_e(e)$ for every e, and
- $f(e) < h(e)$ for every e.

Theorem (Bienvenu, Porter)
DNC_h is a deep class if and only if $\sum_{n=0}^{\infty} \frac{1}{h(n)} = \infty$.

Moreover, if $\sum_{n=0}^{\infty} \frac{1}{h(n)} < \infty$, then every Martin-Löf random computes a DNC_h function.

Thank you for your attention!

Completions of PA proof sketch, 1

To prove that this result, we can consider the class $\mathcal{P}$ of total extensions of a universal partial computable $\{0,1\}$-valued function.

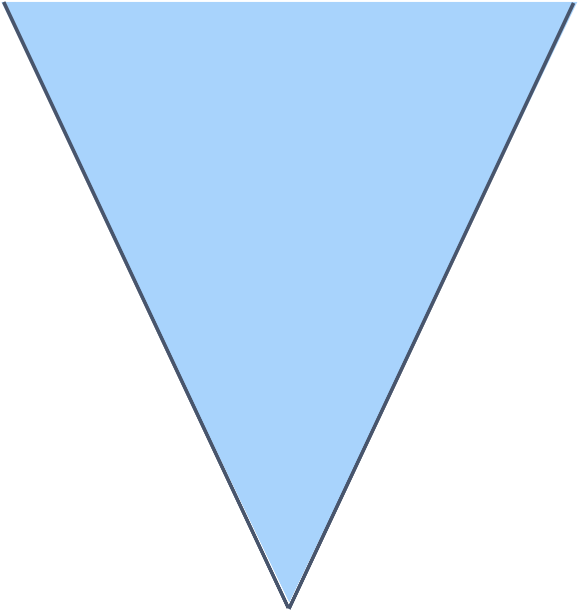
Let $u(\langle e, x\rangle) = \phi_e(x)$, where $(\phi_e)_{e\in\omega}$ is an effective enumeration of all partial computable $\{0,1\}$-valued functions.

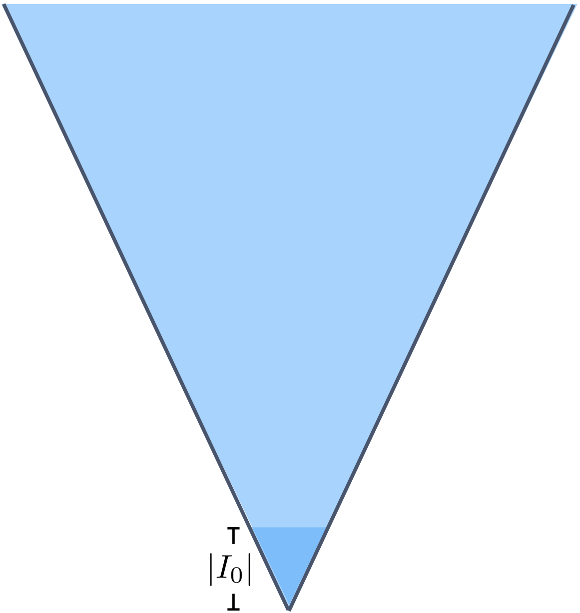
We will define a partial computable $\{0,1\}$-valued function ϕ_e (where we know e in advance by the recursion theorem), and this will allow us to show that $\mathcal{P}$ is deep.

Completions of PA proof sketch, 2

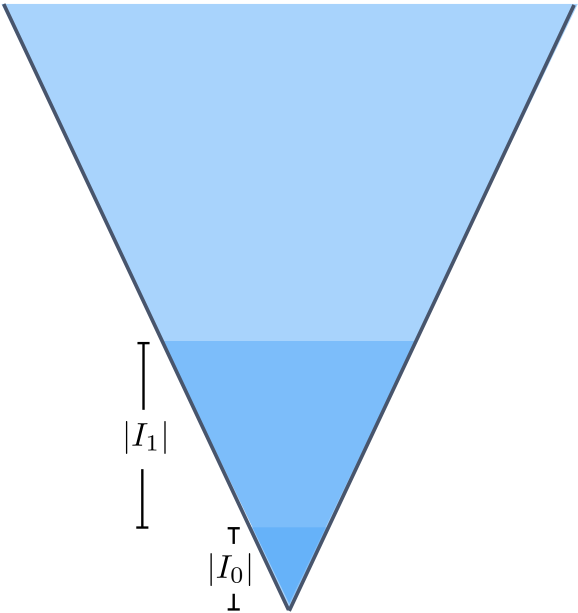
Since we are defining ϕ_e, we have control of the values $u(\langle e, x\rangle)$ for every $x \in \omega$.

Let $(I_k)_{k\in\omega}$ be an effective collection of intervals forming a partition of ω, where we have control of 2^{k+1} values of u inside of I_k for each $k \in \omega$.

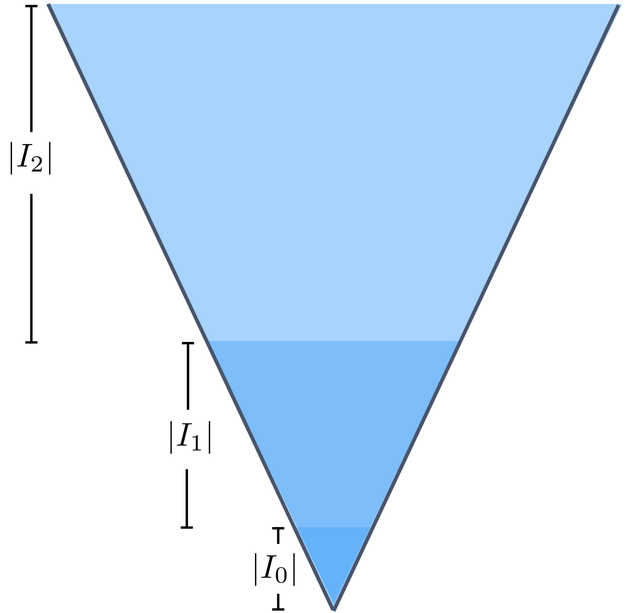


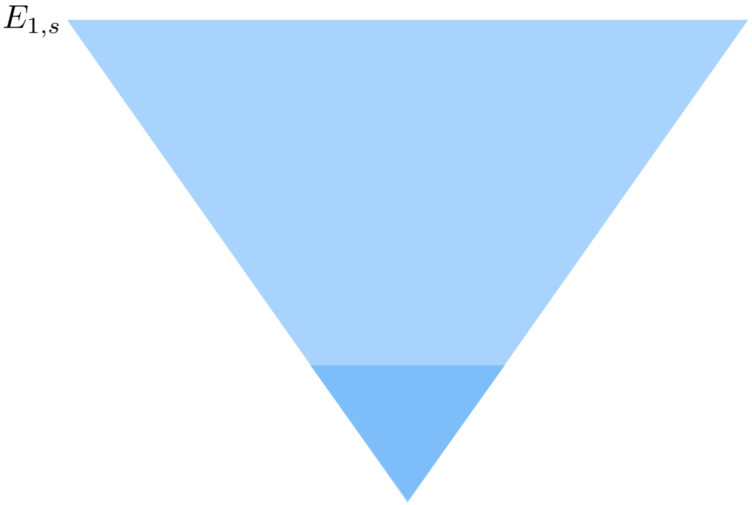


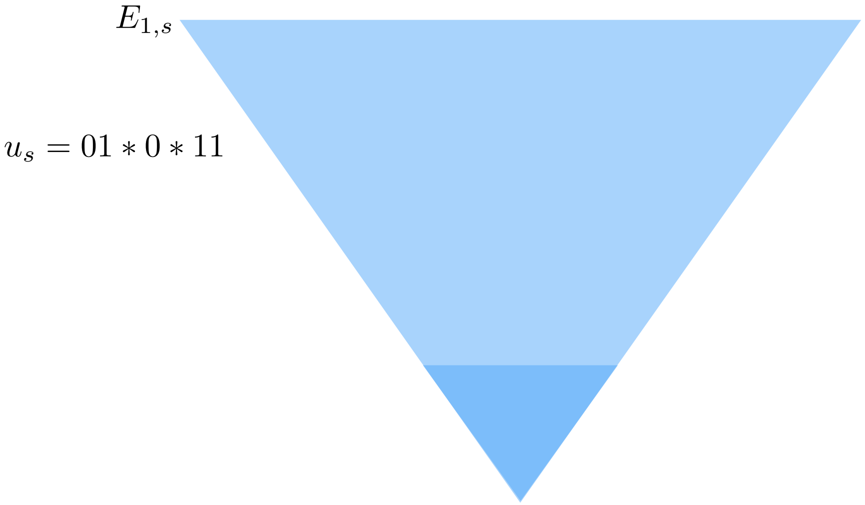
$|I_0|$



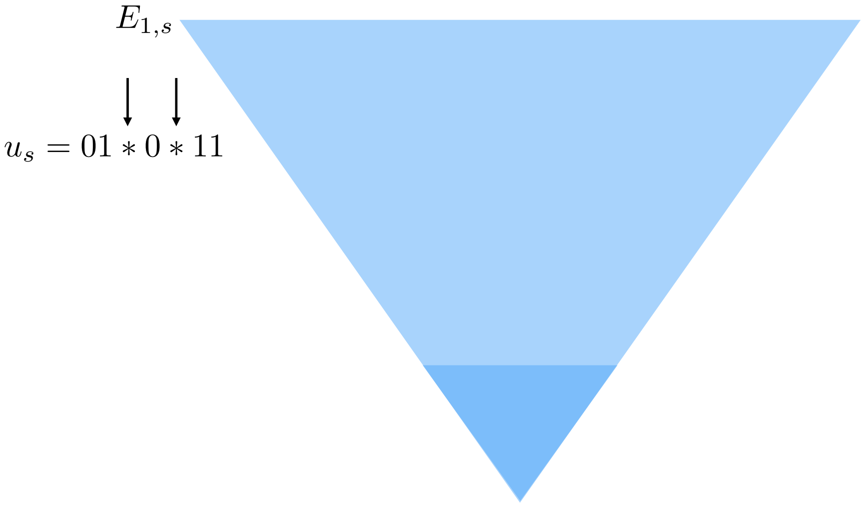
$|I_1|$
$|I_0|$

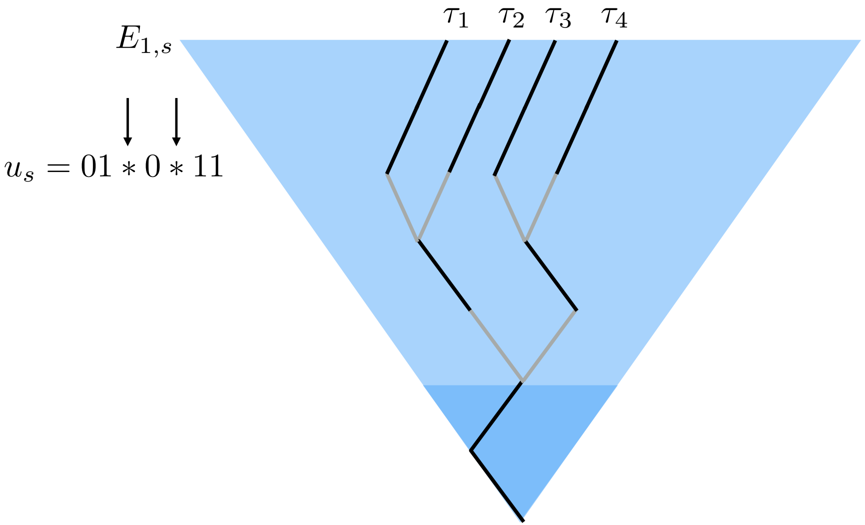

$|I_2|$
$|I_1|$
$|I_0|$


$E_{1,s}$

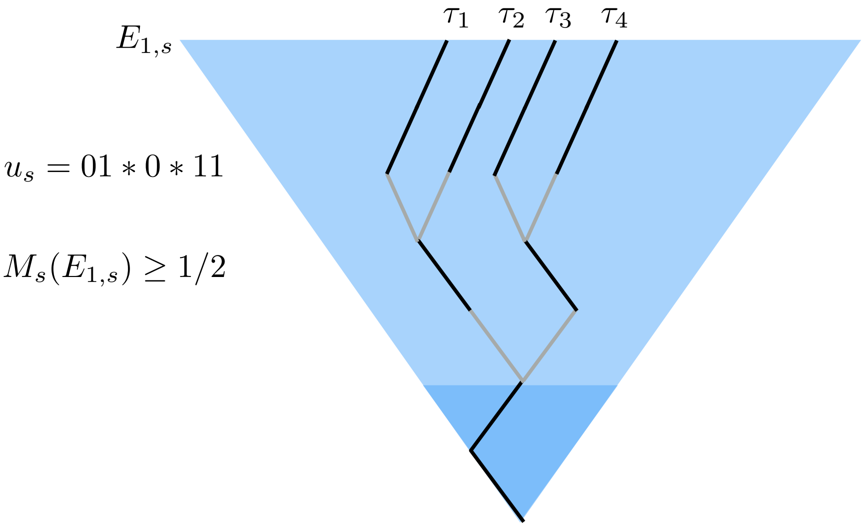


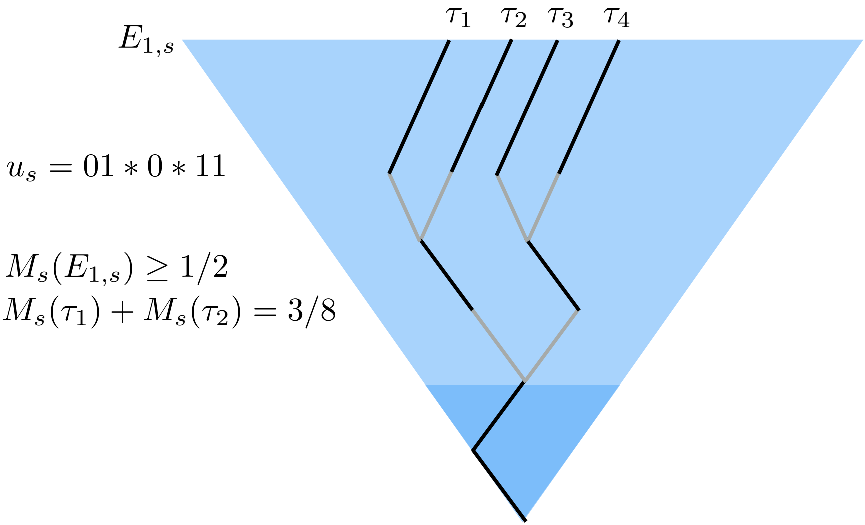
$E_{1,s}$
$u_s = 01 * 0 * 11$

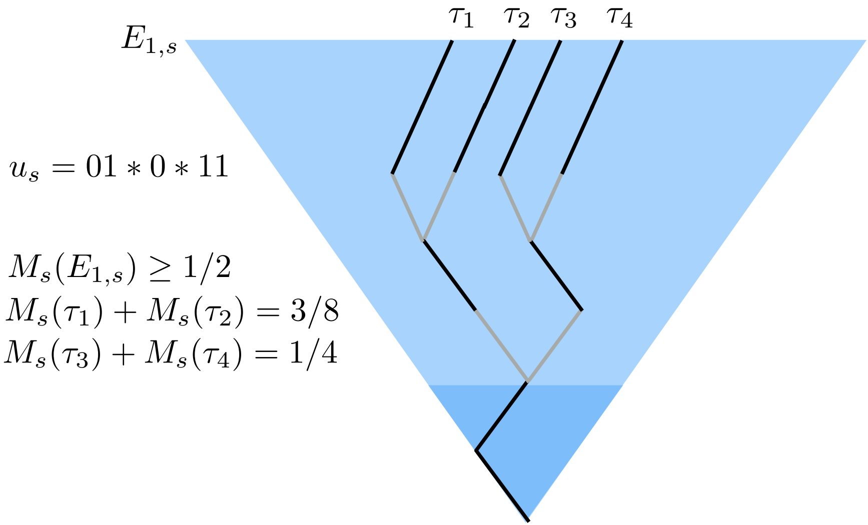

$E_{1,s}$
$u_s = 01 * 0 * 11$



τ_1 τ_2 τ_3 τ_4
$E_{1,s}$
$u_s = 01 * 0 * 11$

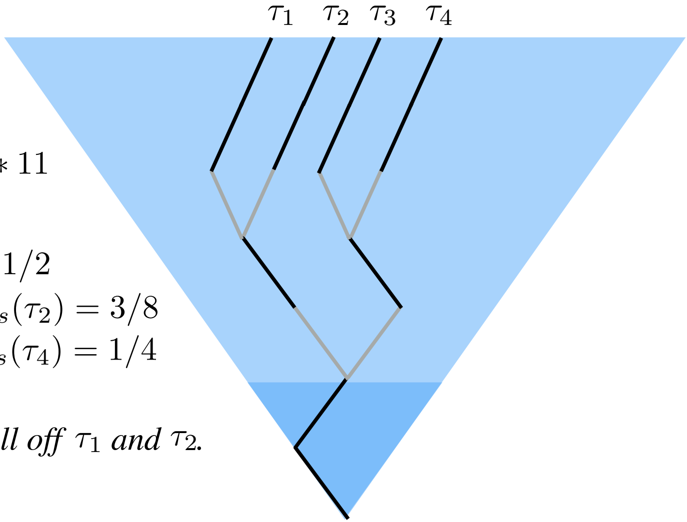

$E_{1,s}$
$u_s = 01 * 0 * 11$
$M_s(E_{1,s}) \geq 1/2$
τ_1 τ_2 τ_3 τ_4


τ_1
τ_2
τ_3
τ_4
$E_{1,s}$
$u_s = 01 * 0 * 11$
$M_s(E_{1,s}) \geq 1/2$
$M_s(\tau_1) + M_s(\tau_2) = 3/8$



τ_1 τ_2 τ_3 τ_4
$E_{1,s}$
$u_s = 01 * 0 * 11$
$M_s(E_{1,s}) \geq 1/2$
$M_s(\tau_1) + M_s(\tau_2) = 3/8$
$M_s(\tau_3) + M_s(\tau_4) = 1/4$

$E_{1,s}$

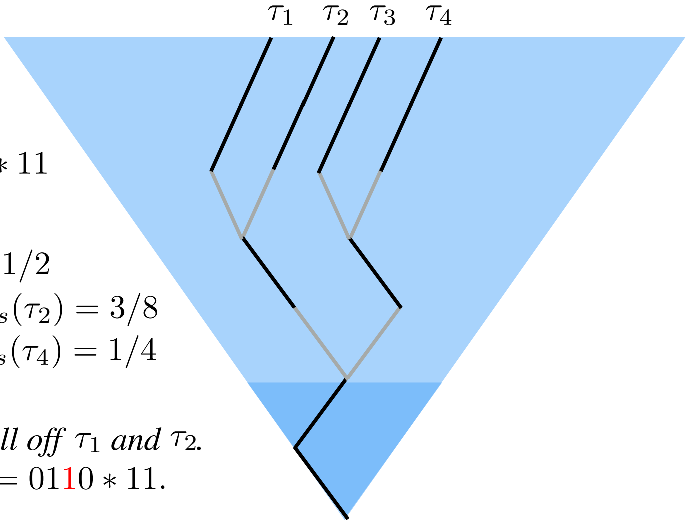


$$u_s = 01 * 0 * 11$$

$$M_s(E_{1,s}) \geq 1/2$$
$$M_s(\tau_1) + M_s(\tau_2) = 3/8$$
$$M_s(\tau_3) + M_s(\tau_4) = 1/4$$

We want to kill off τ_1 and τ_2.

$E_{1,s}$



$u_s = 01 * 0 * 11$

$M_s(E_{1,s}) \geq 1/2$

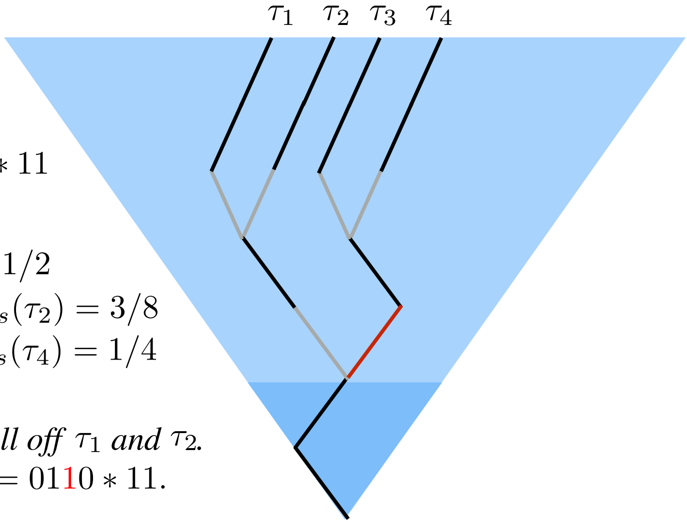
$M_s(\tau_1) + M_s(\tau_2) = 3/8$

$M_s(\tau_3) + M_s(\tau_4) = 1/4$

We want to kill off τ_1 *and* τ_2.

We set $u_{s+1} = 0110 * 11.$

$E_{1,s}$



$u_s = 01 * 0 * 11$

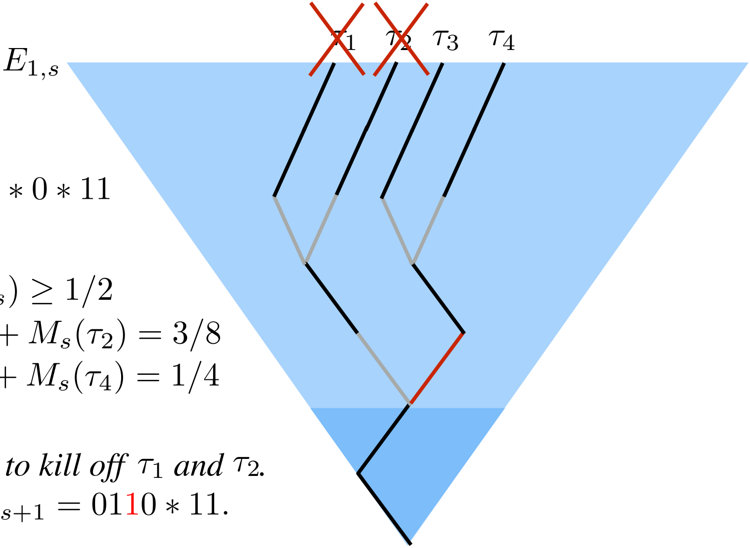
$M_s(E_{1,s}) \geq 1/2$

$M_s(\tau_1) + M_s(\tau_2) = 3/8$

$M_s(\tau_3) + M_s(\tau_4) = 1/4$

We want to kill off τ_1 *and* τ_2.

We set $u_{s+1} = 0110 * 11$.



$u_s = 01 * 0 * 11$

$M_s(E_{1,s}) \geq 1/2$

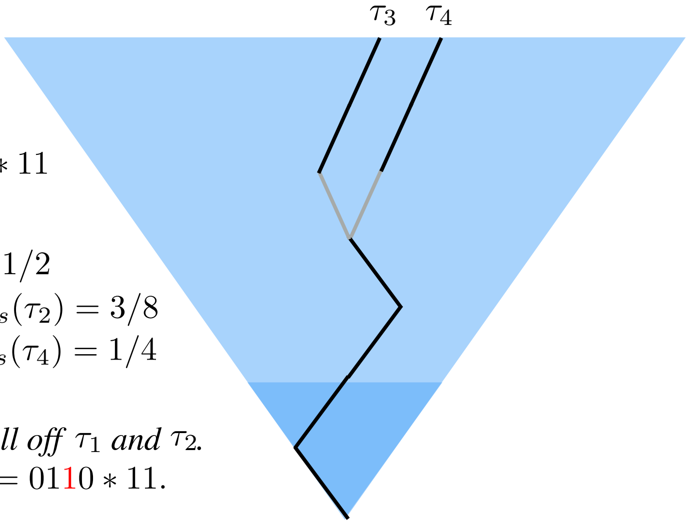
$M_s(\tau_1) + M_s(\tau_2) = 3/8$

$M_s(\tau_3) + M_s(\tau_4) = 1/4$

We want to kill off τ_1 *and* τ_2.

We set $u_{s+1} = 0110 * 11$.

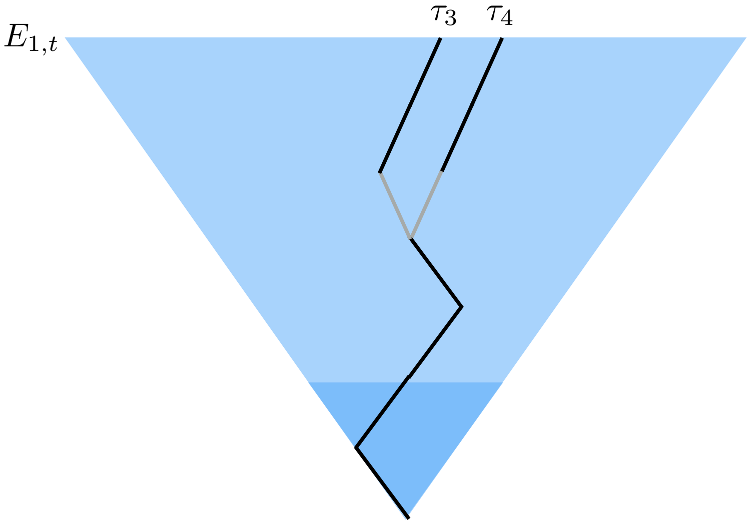
$E_{1,s+1}$



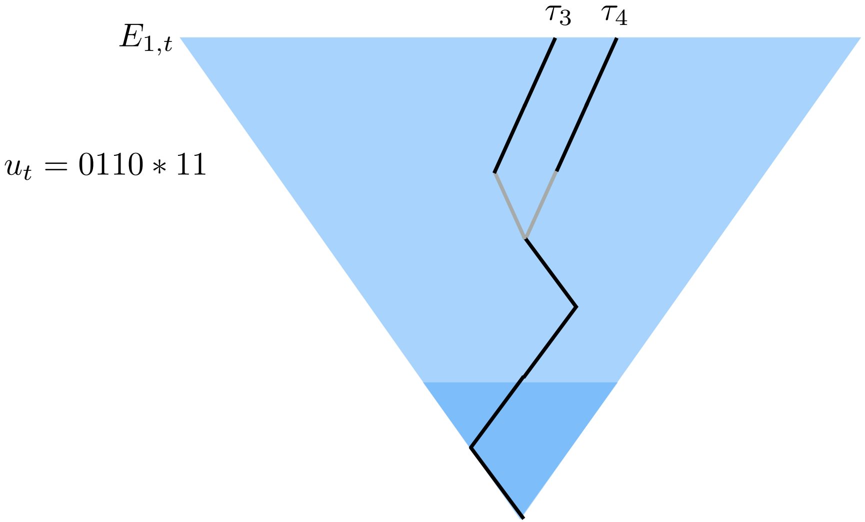
$u_s = 01 * 0 * 11$

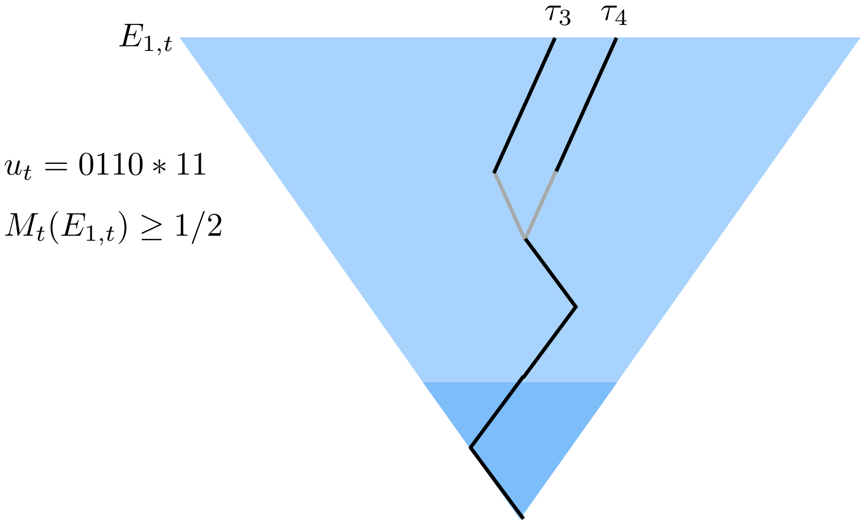
$M_s(E_{1,s}) \geq 1/2$
$M_s(\tau_1) + M_s(\tau_2) = 3/8$
$M_s(\tau_3) + M_s(\tau_4) = 1/4$

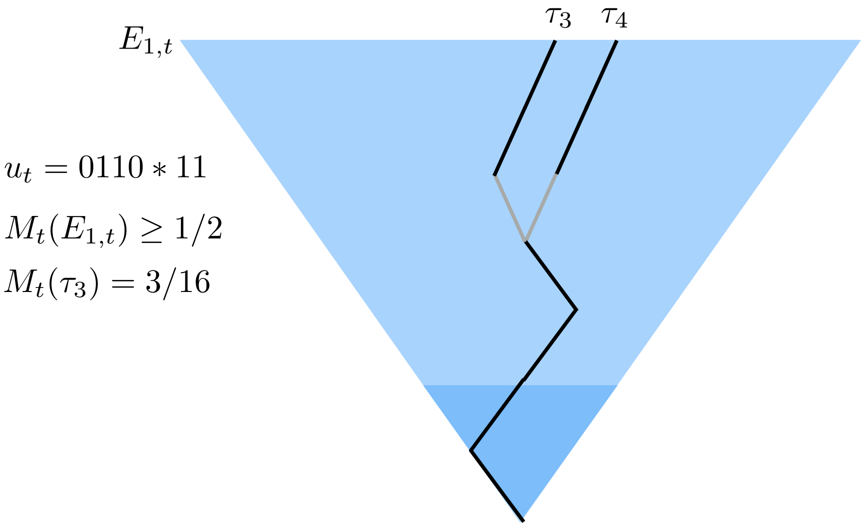
We want to kill off τ_1 *and* τ_2.
We set $u_{s+1} = 0110 * 11$.



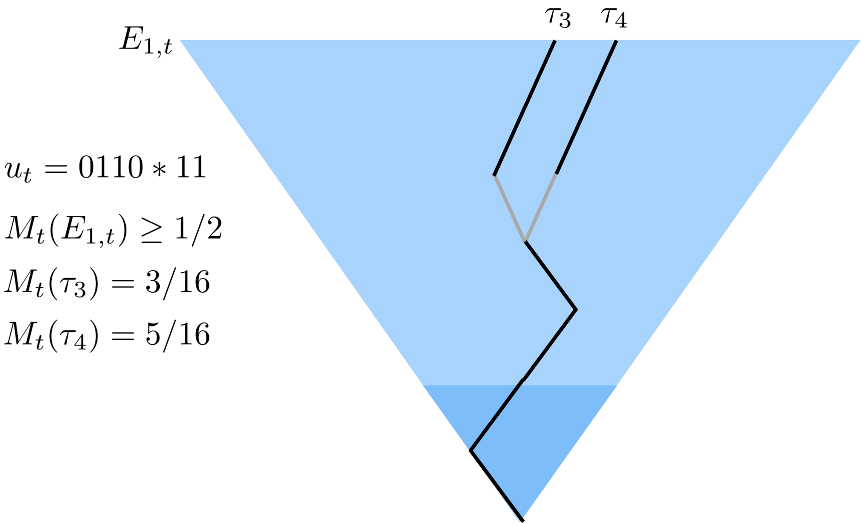
$E_{1,t}$
τ_3
τ_4


τ_3
τ_4
$E_{1,t}$
$u_t = 0110 * 11$

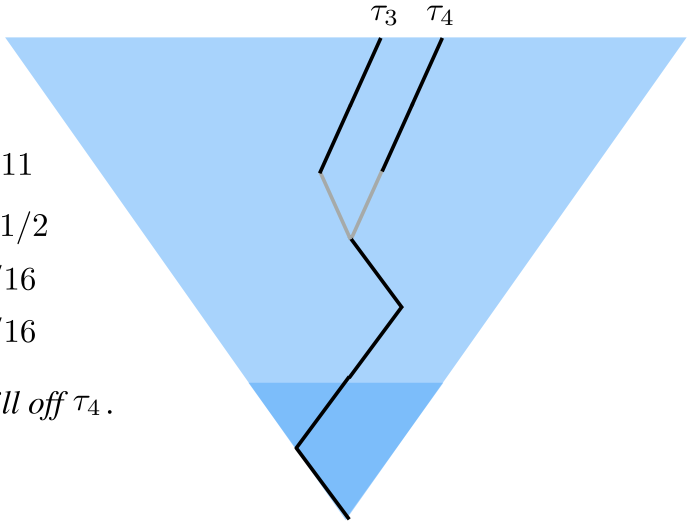

$E_{1,t}$
τ_3
τ_4
$u_t = 0110 * 11$
$M_t(E_{1,t}) \geq 1/2$



τ_3
τ_4
$E_{1,t}$
$u_t = 0110 * 11$
$M_t(E_{1,t}) \geq 1/2$
$M_t(\tau_3) = 3/16$


$E_{1,t}$
τ_3
τ_4
$u_t = 0110 * 11$
$M_t(E_{1,t}) \geq 1/2$
$M_t(\tau_3) = 3/16$
$M_t(\tau_4) = 5/16$

$E_{1,t}$



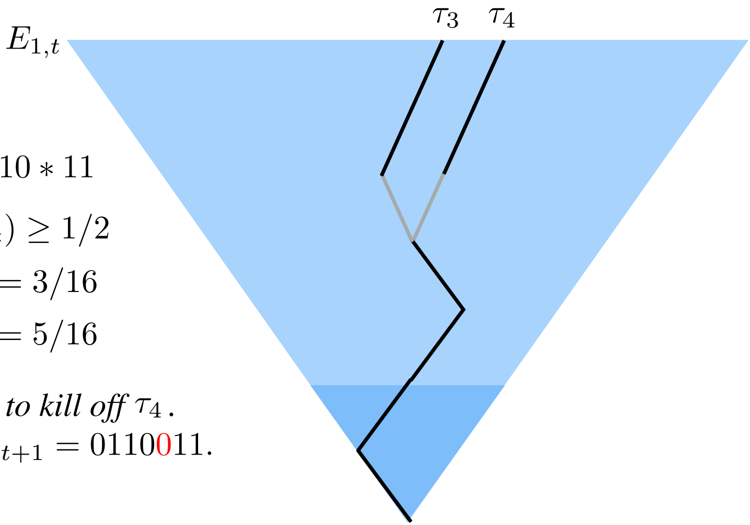
$$u_t = 0110 * 11$$

$$M_t(E_{1,t}) \geq 1/2$$

$$M_t(\tau_3) = 3/16$$

$$M_t(\tau_4) = 5/16$$

We want to kill off τ_4.



$u_t = 0110 * 11$

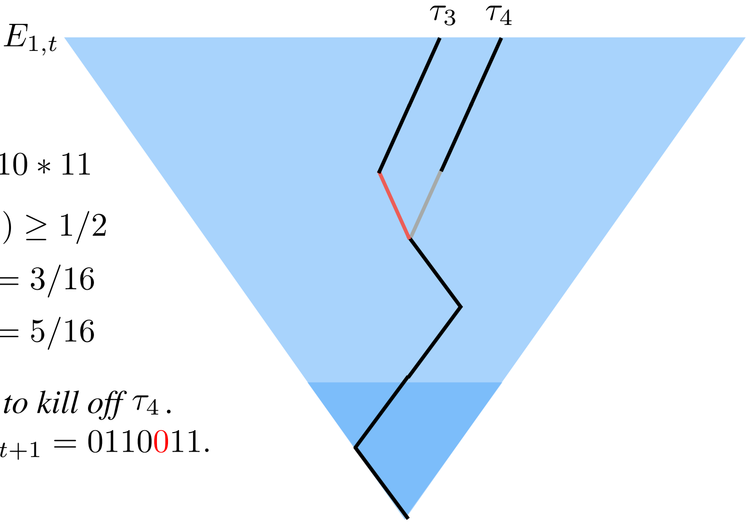
$M_t(E_{1,t}) \geq 1/2$

$M_t(\tau_3) = 3/16$

$M_t(\tau_4) = 5/16$

We want to kill off τ_4.

We set $u_{t+1} = 0110\textcolor{red}{0}11$.



$u_t = 0110 * 11$

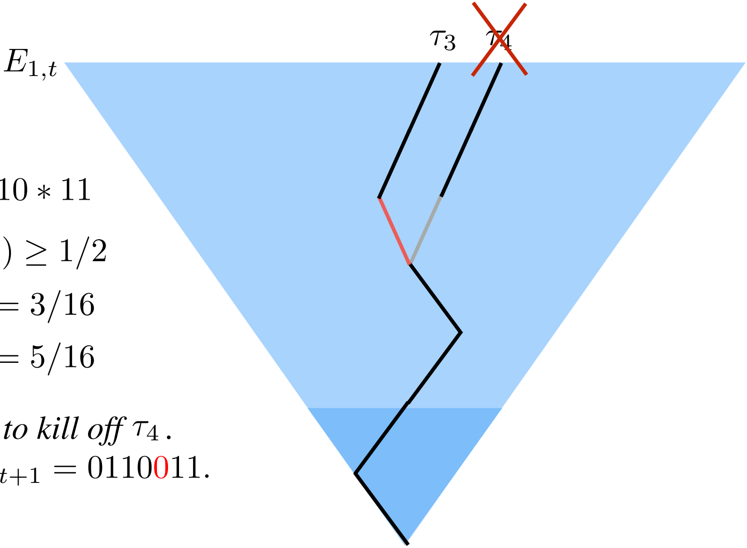
$M_t(E_{1,t}) \geq 1/2$

$M_t(\tau_3) = 3/16$

$M_t(\tau_4) = 5/16$

We want to kill off τ_4.

We set $u_{t+1} = 0110\textcolor{red}{0}11$.



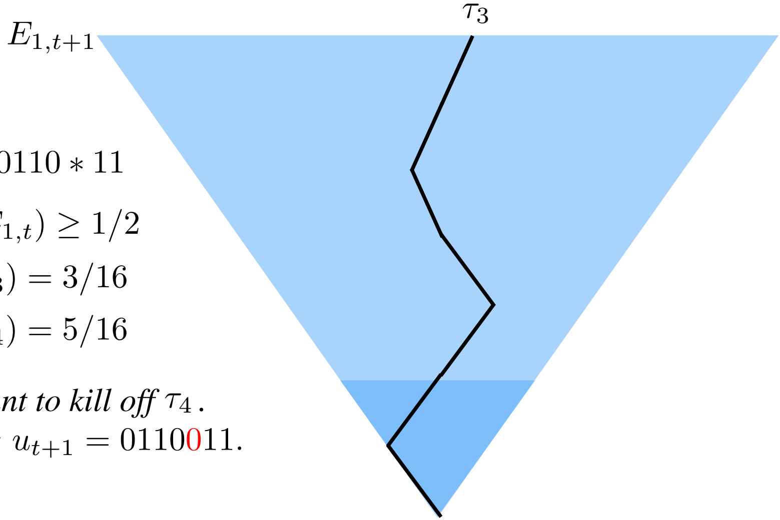
$u_t = 0110 * 11$

$M_t(E_{1,t}) \geq 1/2$

$M_t(\tau_3) = 3/16$

$M_t(\tau_4) = 5/16$

We want to kill off τ_4 .
We set $u_{t+1} = 0110\textcolor{red}{0}11$.



$u_t = 0110 * 11$

$M_t(E_{1,t}) \geq 1/2$

$M_t(\tau_3) = 3/16$

$M_t(\tau_4) = 5/16$

We want to kill off τ_4.
We set $u_{t+1} = 0110\textcolor{red}{0}11$.

Completions of PA proof sketch, 3

Step 1: For each k, we consider the sets

$$E_{k,s} = \{\sigma \in 2^{<\omega} : \sigma \restriction I_k \text{ extends } u_s \restriction I_k\},$$

and wait for a stage s such that

$$M(E_{k,s}) \geq 2^{-k}.$$

Completions of PA proof sketch, 4

Step 2: Pick some $y \in I_k$ on which we have yet to define u.

Consider the sets

$$E^0_{k,s}(y) = \{\sigma \in E_{k,s} : \sigma(y) = 0\}$$

and

$$E^1_{k,s}(y) = \{\sigma \in E_{k,s} : \sigma(y) = 1\}.$$

Then $M(E^i_{k,s}(y)) \geq 2^{-(k+1)}$ for $i = 0$ or 1 (or both).

If this holds for $i = 0$, we set $u(y) = 1$; otherwise we set $u(y) = 0$.

Completions of PA proof sketch, 5

We repeat the process, going back to Step 1.

We can repeat the process at most 2^{k+1} times (since we have enough values to work with in I_k).

Eventually, we will get stuck at Step 1.

Setting $f(k) = \max(I_k)$, we will have

$$M(\{\sigma : \sigma \restriction f(k) \text{ extends } u\}) \leq 2^{-k}.$$

That is,

$$M(T_{f(k)}) \leq 2^{-k}.$$

Establishing the depth of a given Π^0_1 class

The technique for showing that the class of consistent completions of PA is deep is what we refer to as a *wait and kill* argument.

We need to work with some object that we have control over in some way.

For example, in the previous proof we define a partial computable $\{0, 1\}$-valued function ϕ using the recursion theorem.

We *wait* to see a sufficiently large collection of oracles compute some possible extension of ϕ (at some place at which ϕ is currently undefined).

We then define ϕ at this place in such a way as to *kill* off each of these oracles.