The Preservation of Algorithmic Randomness
Part 2

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Outline

1. Review
2. The no randomness ex nihilo principle
3. Applications of randomness preservation

Part 1: Review

Randomness preservation

Theorem

Suppose that Φ is an almost total Turing functional and $X \in \mathsf{MLR}$. Then $\Phi(X) \in \mathsf{MLR}_{\lambda_\Phi}$.

Important facts about Martin-Löf randomness

- For every computable measure μ, there is a universal μ-Martin-Löf test.
- If μ is a computable measure, then no Martin-Löf random sequence is contained in a μ-null Π^0_1 class.
- $X \oplus Y \in \mathrm{MLR}$ if and only if $X \in \mathrm{MLR}^Y$ and $Y \in \mathrm{MLR}$.

Part 2: The no randomness ex nihilo principle

A question from last time

Question

If Φ is an almost total Turing functional and $Y \in \mathrm{MLR}_{\lambda_\Phi}$, is it the case that $\Phi^{-1}(\{Y\}) \subseteq \mathrm{MLR}$?

In general, the answer is NO.

Let Φ be defined by $\Phi(X \oplus Y) = Y$.

Note that for any $Y \in \mathrm{MLR}$ we can always some $X \oplus Y \in \mathrm{MLR}$ such that $\Phi(X \oplus Y) = Y$.

No randomness ex nihilo

Theorem

Suppose that Φ is an almost total Turing functional and $Y \in \mathrm{MLR}_{\lambda_\Phi}$. Then there is some $X \in \mathrm{MLR}$ such that $\Phi(X) = Y$.

Proof

Let $(\mathcal{U}_i)_{i\in\omega}$ be a universal Martin-Löf test with respect to the Lebesgue measure.

Let $\mathcal{K}_n = 2^\omega \setminus \mathcal{U}_n$.

Claim: $\mathcal{K}_n \subseteq \mathrm{dom}(\Phi)$ for every $n \in \omega$.

$\mathrm{dom}(\Phi) = \bigcap_{i\in\omega}\{X : |\Phi(X)| \geq i\}$, which is Π^0_2.

Since $\lambda(\mathrm{dom}(\Phi)) = 1$, $2^\omega \setminus \mathrm{dom}(\Phi)$ is a Σ^0_2 null set.

Thus, if $X \notin \mathrm{dom}(\Phi)$, then X is contained in a Π^0_1 null set, which implies that $X \notin \mathrm{MLR}$.

Since $\mathcal{K}_n \subseteq \mathrm{MLR}$, the claim follows.

Proof (continued)

$\mathcal{K}_n \subseteq \mathrm{dom}(\Phi)$ implies that $\Phi(\mathcal{K}_n)$ is a Π^0_1 class.

In fact, the collection $(\Phi(\mathcal{K}_n))_{n\in\omega}$ is uniformly Π^0_1.

Hence, the collection $(2^\omega \setminus \Phi(\mathcal{K}_n))_{n\in\omega}$ is uniformly Σ^0_1.

Then

$$\begin{aligned}\lambda_\Phi(2^\omega \setminus \Phi(\mathcal{K}_n)) &= 1 - \lambda_\Phi(\Phi(\mathcal{K}_n))\\ &= 1 - \lambda(\Phi^{-1}(\Phi(\mathcal{K}_n)))\\ &\leq 1 - \lambda(\mathcal{K}_n) \leq 2^{-n}.\end{aligned}$$

Thus $(2^\omega \setminus \Phi(\mathcal{K}_n))_{n\in\omega}$ is a λ_Φ-Martin-Löf test.

Proof (continued)

$(2^\omega \setminus \Phi(\mathcal{K}_n))_{n\in\omega}$ is a λ_Φ-Martin-Löf test.

Suppose that $Y \in \mathrm{MLR}_{\lambda_\Phi}$.

Then there is some n such that $Y \notin 2^\omega \setminus \Phi(\mathcal{K}_n)$

Hence $Y \in \Phi(\mathcal{K}_n)$.

Since $\mathcal{K}_n \subseteq \mathrm{MLR}$, it follows that $\Phi^{-1}(Y) \cap \mathrm{MLR} \neq \emptyset$.

Question about almost totality

Question

Do we still have randomness preservation if we weaken the condition of almost totality?

In general, the answer is NO.

Theorem (V'yugin)

For every $\epsilon > 0$, there is a Turing functional Φ_ϵ with $\lambda(\mathrm{dom}(\Phi_\epsilon)) > 1 - \epsilon$ such that for every $X \in \mathrm{MLR} \cap \mathrm{dom}(\Phi_\epsilon)$,

- *$\Phi_\epsilon(X)$ is not random with respect to any computable measure, and*
- *$\Phi_\epsilon(X)$ cannot even compute any non-computable sequence that is random with respect to some computable measure.*

Part 3: Applications of randomness preservation

Interplay between classes of algorithmically random objects

In recent work with Quinn Culver, we studied the interactions between algorithmically random members of

- $\mathcal{G}(2^\omega)$, the set of closed subsets of 2^ω,
- $\mathcal{C}(2^\omega)$, the set of continuous functions on 2^ω, and
- $\mathcal{P}(2^\omega)$, the set of probability measures on 2^ω.

Random closed sets

One way to define an algorithmically random closed subset of 2^ω:

- A closed set $\mathcal{C} \subseteq 2^\omega$ is random if it can be coded by an algorithmically random sequence $X \in 3^\omega$ as shown by the following example.

This definition is due to Barmpalias, Brodhead, Dashti, Cenzer, and Weber.

$X = 2021122012\ldots$

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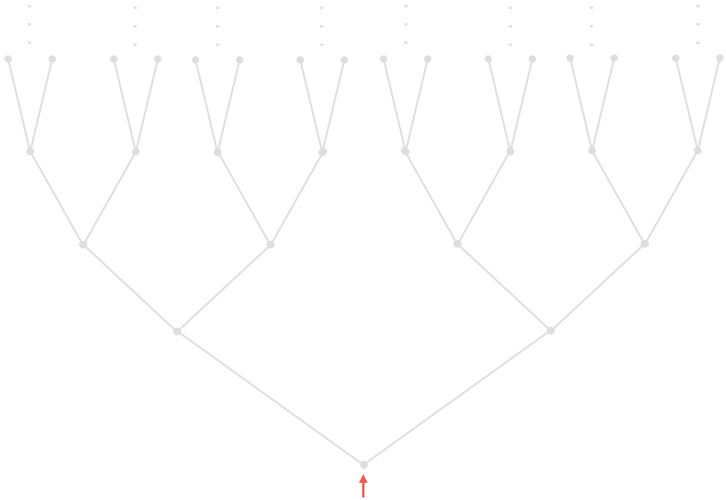

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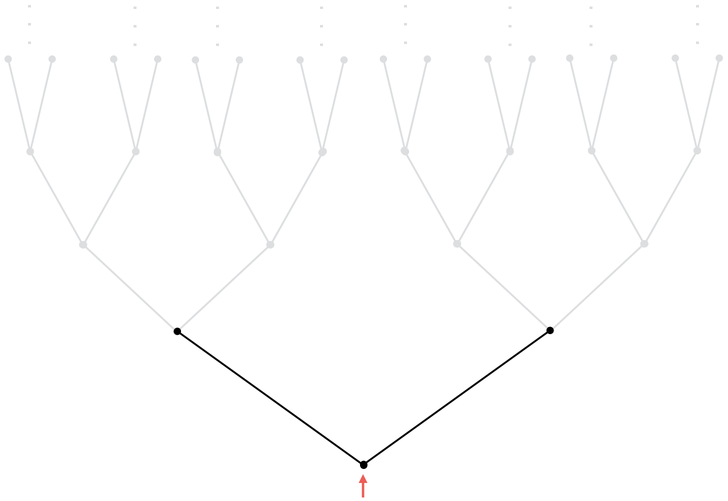

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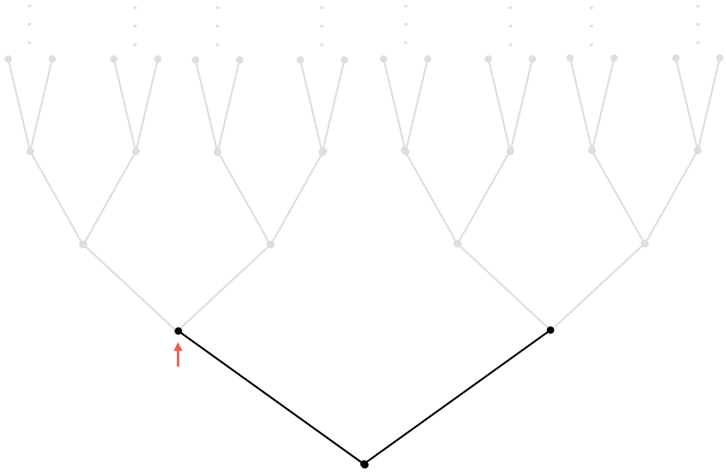
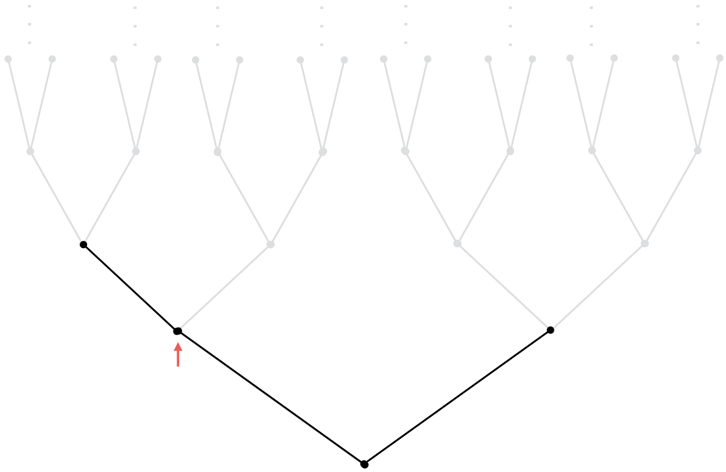
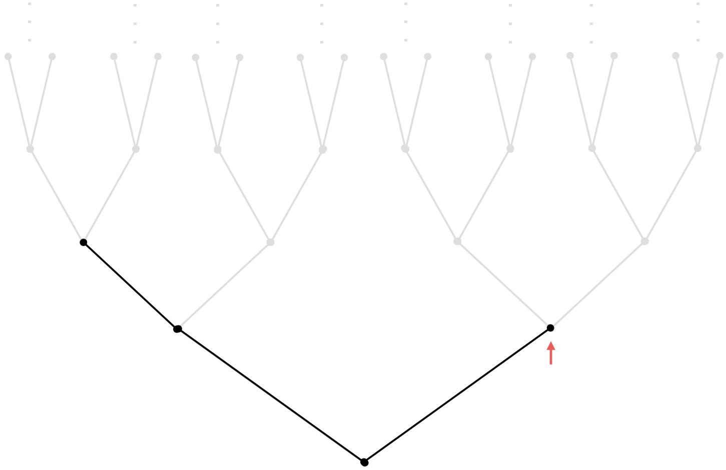


$X = 2\textcolor{red}{0}21122012\ldots$



$X = 2\textcolor{red}{0}21122012\ldots$



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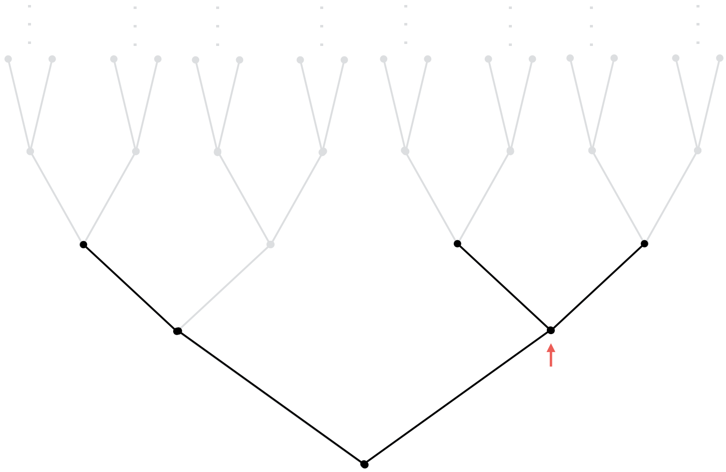

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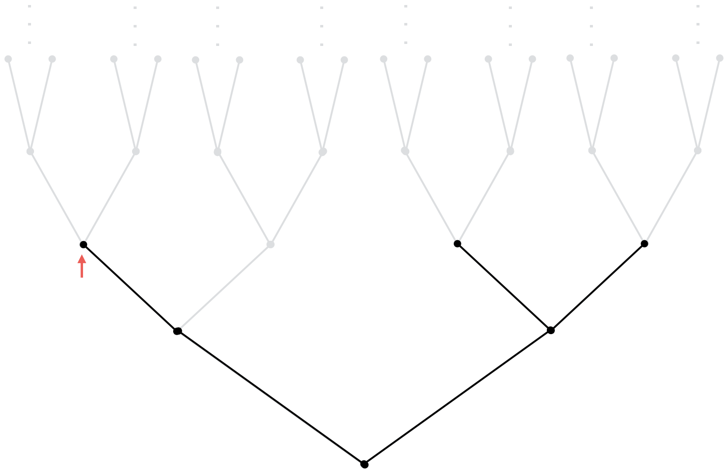
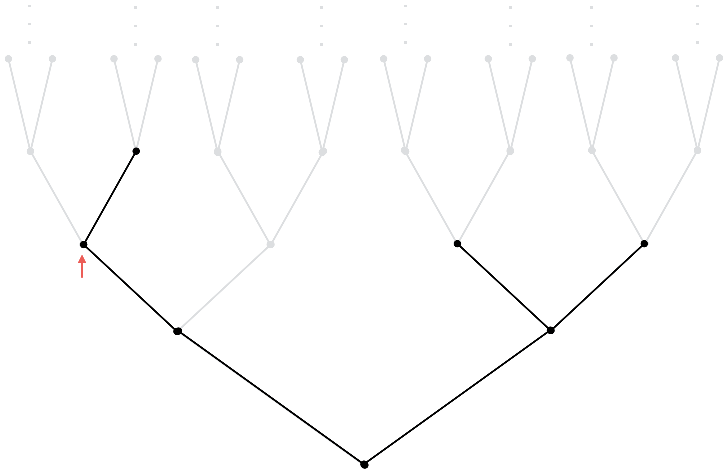
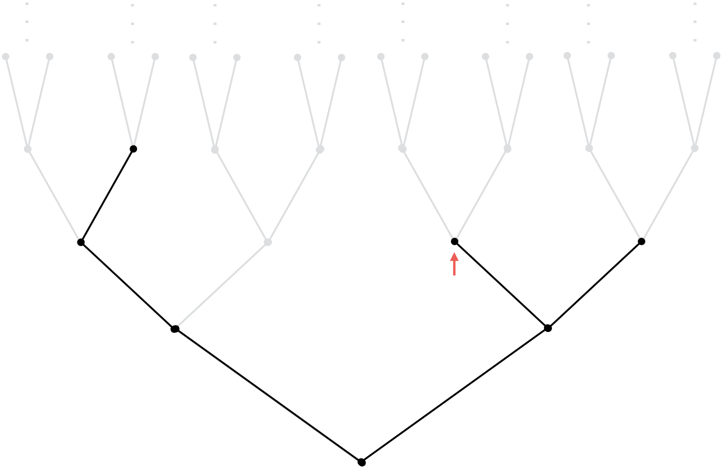


$$X = 2021122012\ldots$$



$$X = 2021122012\ldots$$



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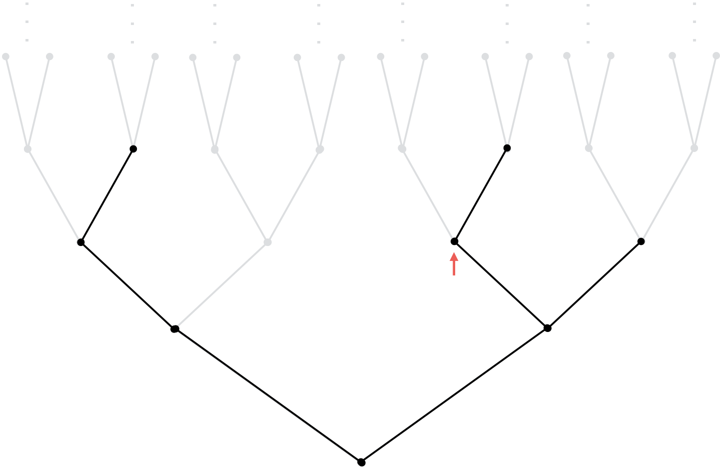
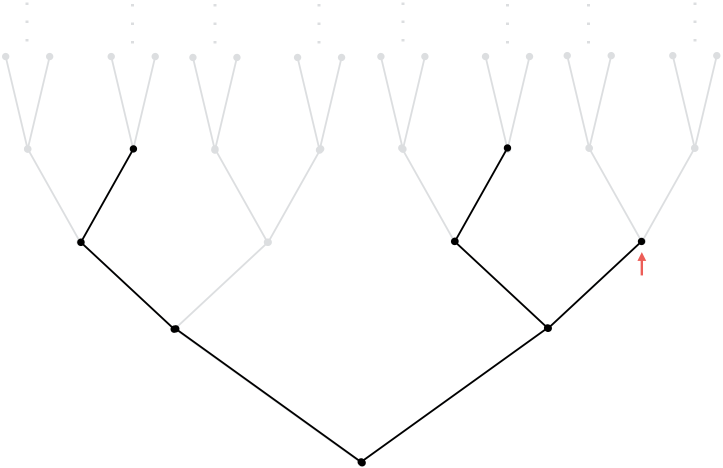


$X = 2021\color{red}{1}22012\ldots$



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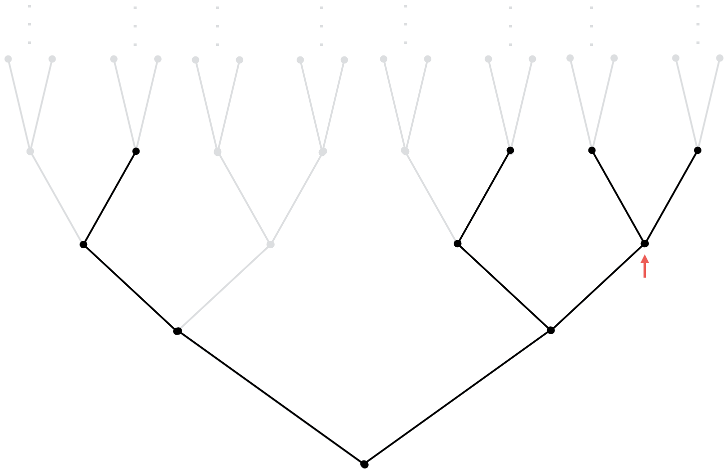

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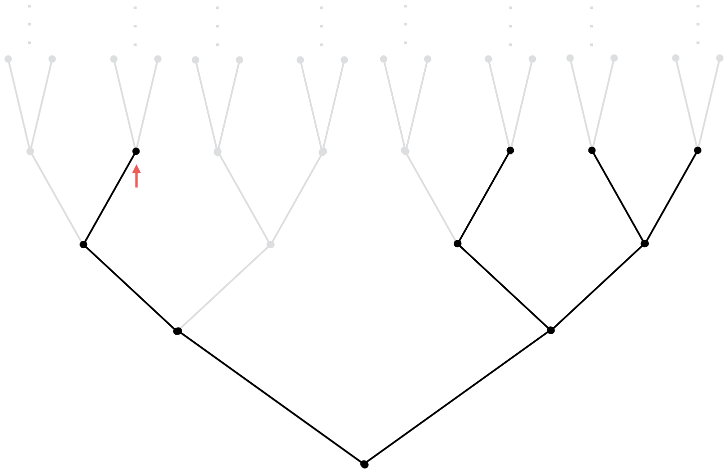

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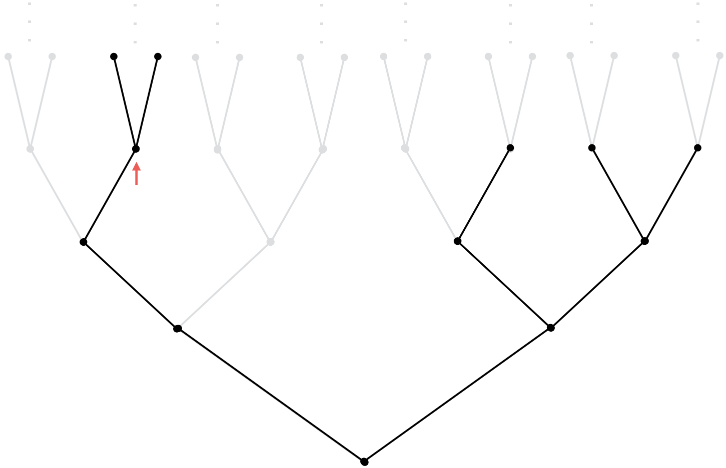

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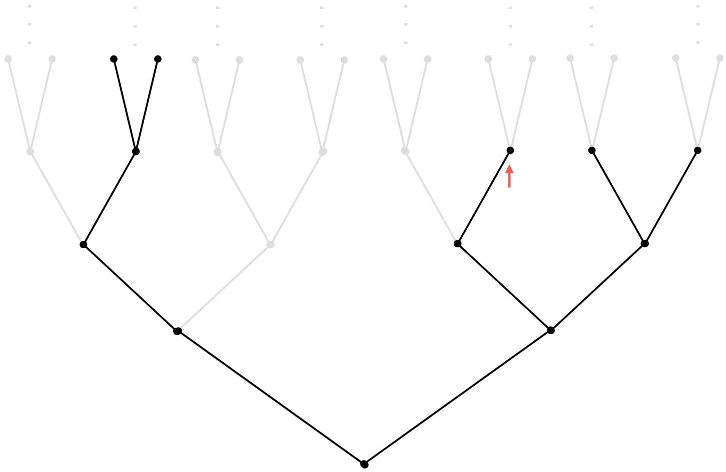
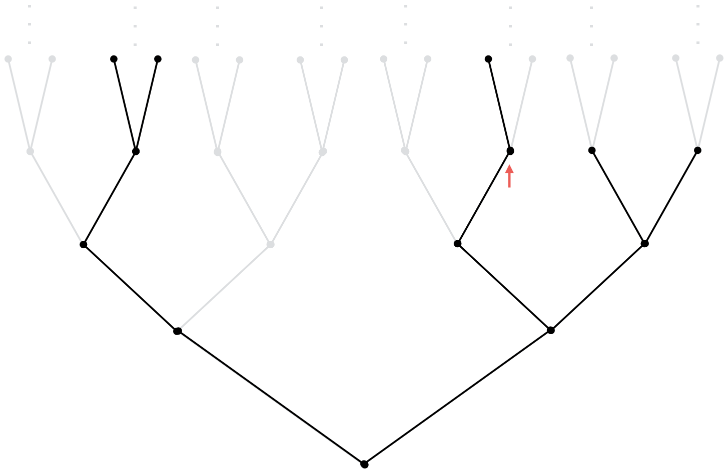
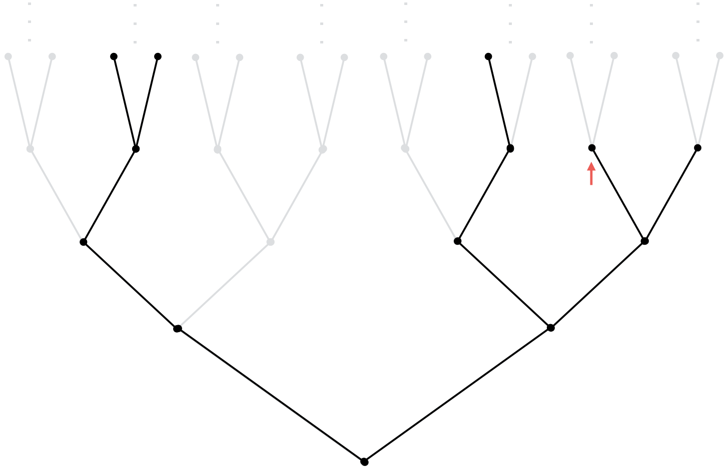


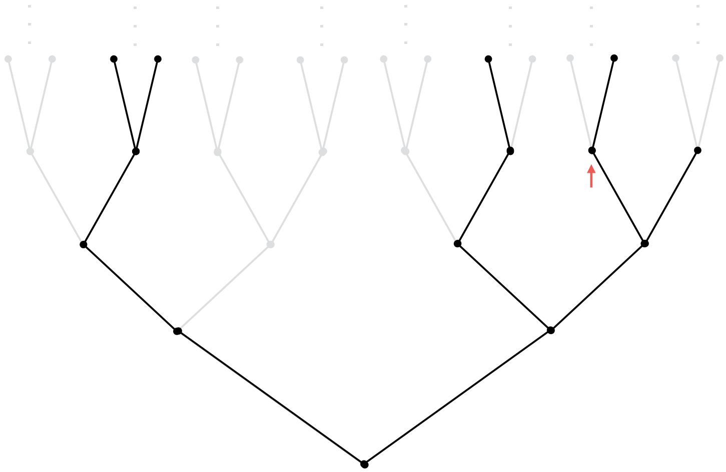
$X = 2021122\color{red}{0}12\ldots$



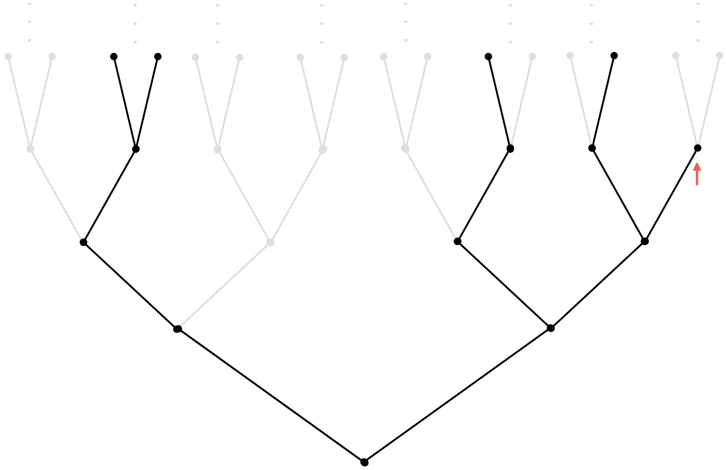
$X = 2021122\textcolor{red}{0}12\ldots$



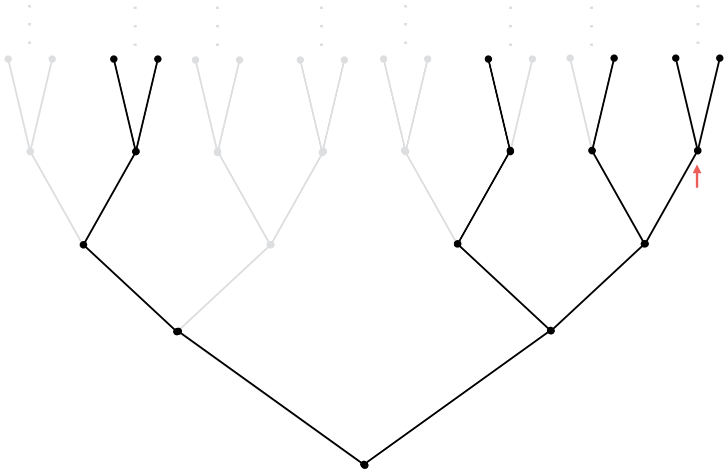
$X = 2021122012\ldots$



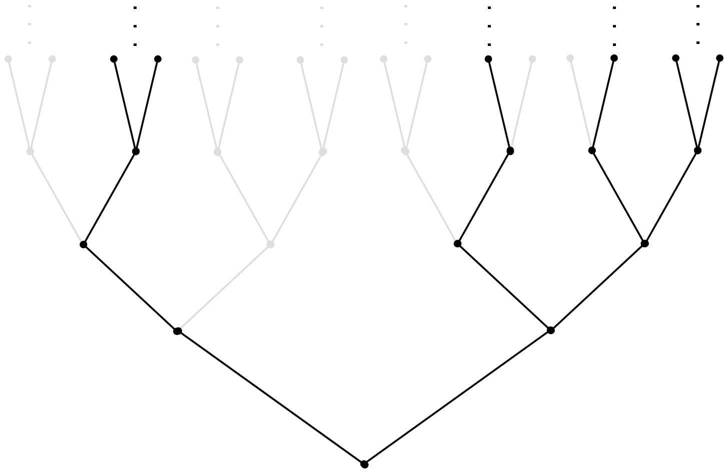
$X = 202112201 2\ldots$



$X = 2021122012\ldots$



$X = 2021122012\ldots$



$X = 2021122012\ldots$

Robustness of this definition

The resulting definition of randomness is equivalent to one obtained by

- defining Martin-Löf random closed sets in a way that is "native" to $\mathcal{G}(2^\omega)$, which uses the Fell topology on 2^ω and Choquet capacity (Axon); and
- defining Martin-Löf random closed sets in terms of a certain Galton-Watson process (Diamondstone, Kjos-Hanssen).

Random paths of random closed sets

Theorem (BBDCW)

Every random closed set contains a random sequence, and every random sequence is contained in some random closed set.

Proof sketch:

- Define a map Φ that sends a random pair $(X, Y) \in 2^\omega \times 3^\omega$ to a path Z in the closed set $\mathcal{C}_Y$ coded by Y.
- Use X as advice to define the path whenever we reach a branching node in $\mathcal{C}_Y$.
- Check: this map induces the Lebesgue measure.
- The result follows from randomness preservation and no randomness ex nihilo.

A follow-up question

Question

Is every member of a random closed set a random sequence?

Answer: NO.

- The map that sends a random closed set $\mathcal{C} \subseteq 2^\omega$ to its leftmost path is a total Turing functional that induces the Bernoulli $(2/3, 1/3)$-measure.
- By randomness preservation, the leftmost path of a random closed set is random with respect to the Bernoulli $(2/3, 1/3)$-measure.

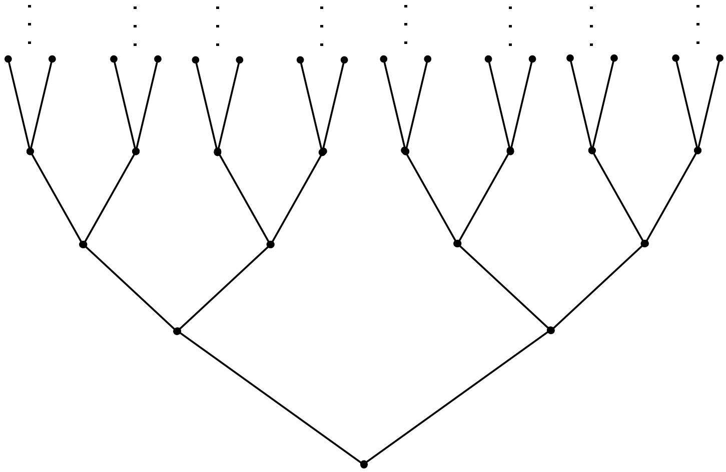
Random continuous functions

One way to define an algorithmically random continuous function on 2^ω:

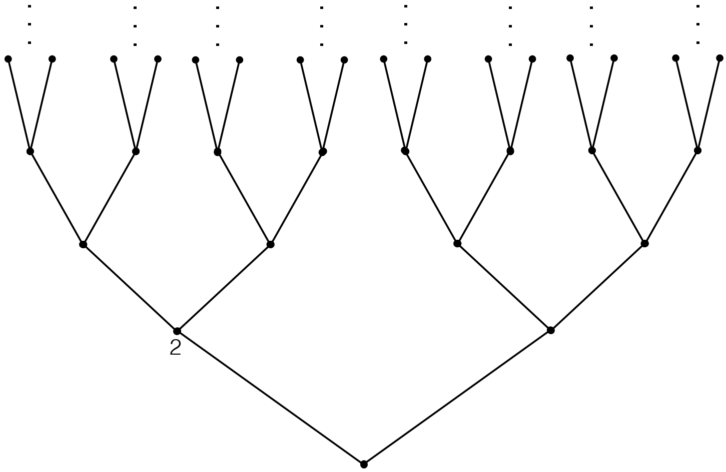
- A continuous function $\mathcal{P} \subseteq 2^\omega$ is random if it can be coded by an algorithmically random sequence $X \in 3^\omega$ as shown by the following example.

This definition is due to Barmpalias, Brodhead, Cenzer, Remmel, and Weber.

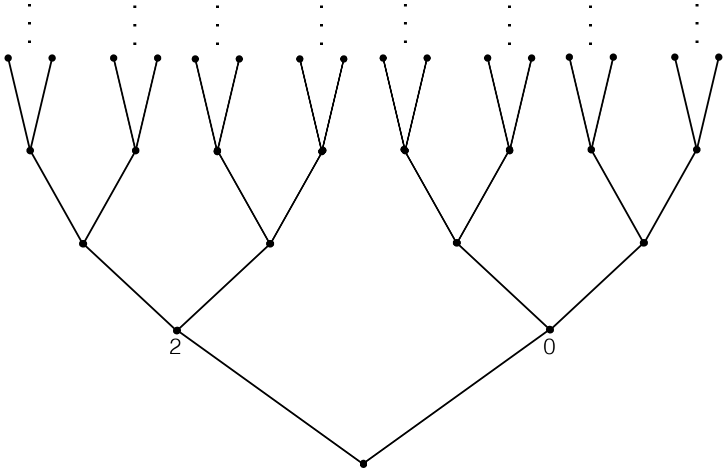
$$X = 2021122012110220011021112112220\ldots$$



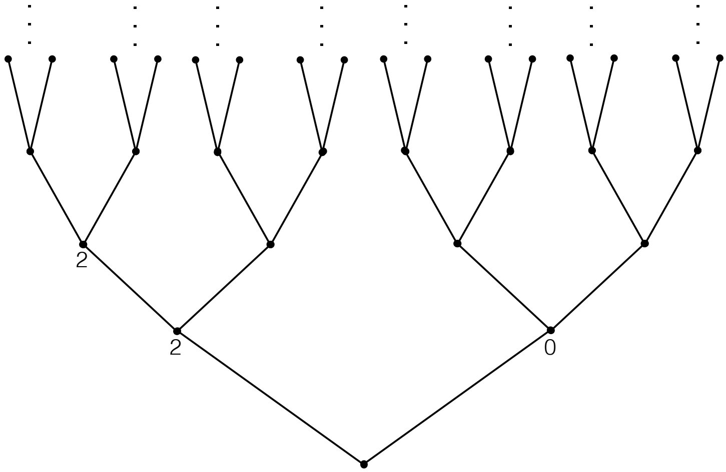
$X = 202112201211022001102111211220\ldots$



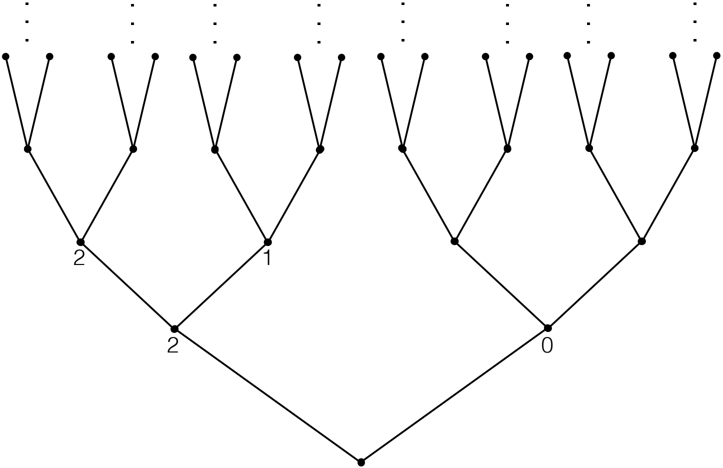
$$X = 202112201211022001102111211220\ldots$$



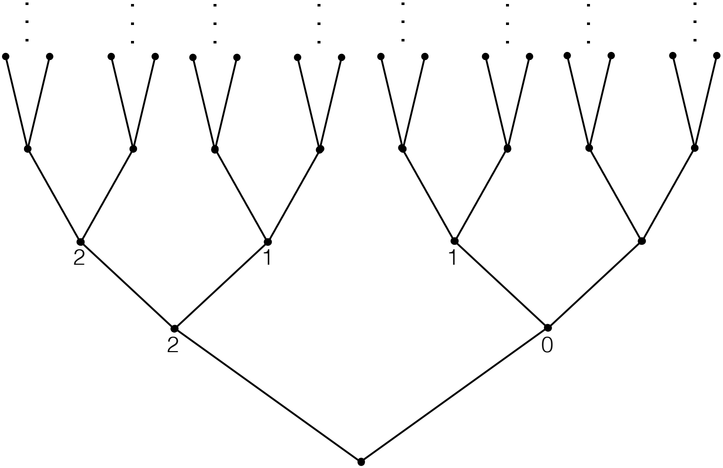
$$X = 2021122012110220011021112112 20\ldots$$



$X = 2021122012110220011021112111220\ldots$



$X = 2021122012110220011021112112220\ldots$



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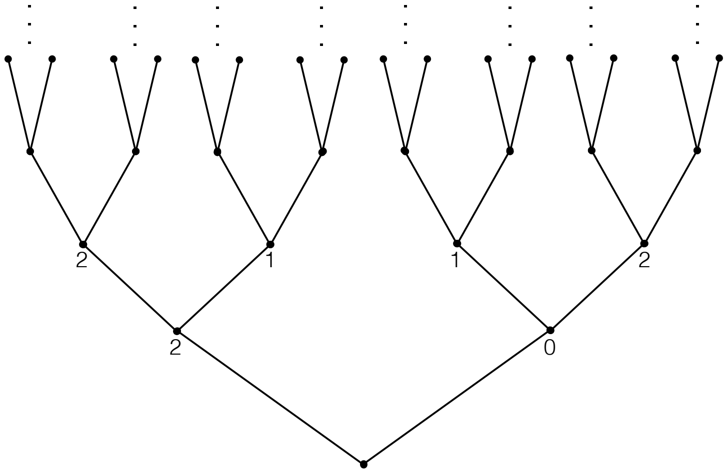
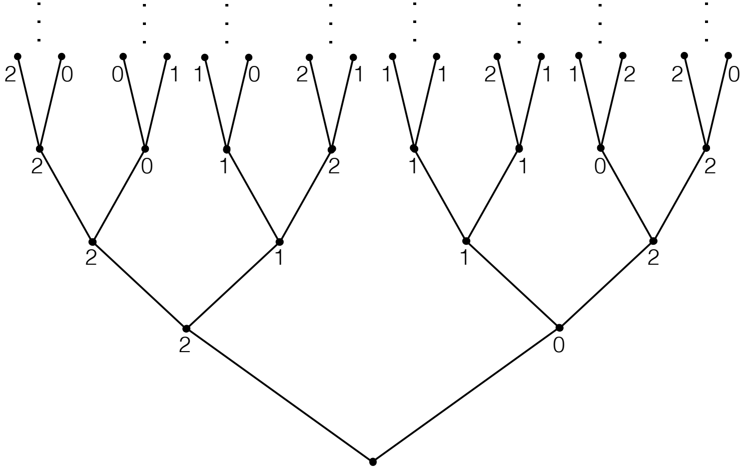


$$X = 202112201211022001102111211220\ldots$$



$X = 202112201211022001102111211220\ldots$

The image of computable point

Proposition (BBCRW)

If $F \in \mathcal{C}(2^\omega)$ is random, then $F(0^\omega) \in \mathrm{MLR}$.

The result still holds if we replace 0^ω with any computable point X.

Proof idea: The map that sends $Z \in 3^\omega$ to $F_Z(0^\omega)$, where F_Z is the function coded by Z, induces the Lebesgue measure.

Computable points in the range?

Question

Is 0^ω in the range of a random continuous function?

Theorem (BBCRW)

For every $Y \in 2^\omega$, $\lambda(\{Z : Y \in \mathrm{ran}(F_Z)\}) = \frac{3}{4}$.

Theorem (BBCRW)

If F *is a random function and* $0^\omega \in \mathrm{ran}(F)$, *then* $F^{-1}(\{0^\omega\})$ *is a random closed set.*

The result holds if we replace 0^ω with any computable sequence.

The converse was left open.

The converse via no randomness ex nihilo

Theorem (Culver-Porter)

For every random closed set $\mathcal{C} \subseteq 2^\omega$, there is a random continuous function F such that $F^{-1}(\{0^\omega\}) = \mathcal{C}$.

A nice corollary

Corollary

The collection of random functions is not closed under composition.

We show that for every random F, there is a random G such that $G \circ F$ is not random.

- Given random $F \in \mathcal{C}(2^\omega)$, there is some $R \in \mathrm{MLR}$ such that $F(0^\omega) = R$.
- By the BBCDW result, R is a member of some random closed set $\mathcal{P}$.
- By the previous theorem, there is a random function G such that $\mathcal{P} = G^{-1}(\{0^\omega\})$, so that $G(R) = 0^\omega$.
- Since $(G \circ F)(0^\omega) = G(R) = 0^\omega$, $G \circ F$ cannot be random.

A question about injectivity

If 0^ω is in the range of a random function F, then F is not injective.

Theorem (Culver-Porter)

No random continuous function is injective.

To prove this, we first show:

Theorem (Culver-Porter)

If $F \in \mathcal{C}(2^\omega)$ *is random, then* $\lambda(\mathrm{ran}(F)) > 0$.

Proof sketch

Theorem (Culver-Porter)

If $F \in \mathcal{C}(2^\omega)$, then $\lambda(\mathrm{ran}(F)) > 0$.

Proof sketch:

- Given a random $F \in \mathcal{C}(2^\omega)$, suppose that $\lambda(\mathrm{ran}(F)) = 0$.
- Since F is total, $\mathrm{ran}(F)$ is $\Pi^{0,F}_1$.
- By the preservation of randomness relative to F, $F(X) \in \mathrm{MLR}^F_{\lambda_F}$ for any $X \in \mathrm{MLR}^F$.
- λ_F is an algorithmically random measure.
- From work of Hoyrup we have

$$\mathrm{MLR} = \bigcup_{F \text{ random}} \mathrm{MLR}_{\lambda_F}.$$

Proof sketch

Theorem (Culver-Porter)

If $F \in \mathcal{C}(2^\omega)$, then $\lambda(\mathrm{ran}(F)) > 0$.

Proof sketch:

- Given a random $F \in \mathcal{C}(2^\omega)$, suppose that $\lambda(\mathrm{ran}(F)) = 0$.
- Since F is total, $\mathrm{ran}(F)$ is $\Pi^{0,F}_1$.
- By the preservation of randomness relative to F, $F(X) \in \mathrm{MLR}^F_{\lambda_F}$.
- λ_F is an algorithmically random measure.
- With some work, one can show

$$\mathrm{MLR}^F = \bigcup_{F \text{ random}} \mathrm{MLR}^F_{\lambda_F}.$$

- Thus $F(X) \in \mathrm{MLR}^F$, which contradicts the fact that $F(X)$ is contained in a $\Pi^{0,F}_1$ null class.

Proof of the injectivity theorem

Theorem (Culver-Porter)

No random continuous function is injective.

Proof sketch:

- Given a random $F \in \mathcal{C}(2^\omega)$, since $\lambda(\mathrm{ran}(F)) > 0$, there is some $Z \in \mathrm{MLR}^F \cap \mathrm{ran}(F)$.
- By relativizing our earlier theorem, $F^{-1}(\{Z\})$ is an F-random closed set, which is perfect.
- Thus F is not injective.