Algorithmic Randomness and Non-Uniform Probability Measures Part 2

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UF Logic Seminar
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Last time

Theorem (Demuth)

If $A \in \mathrm{MLR}$ and Φ is a truth-table functional, if $\Phi(A)$ is not computable, then $\Phi(A) \equiv_T B$ for some $B \in \mathrm{MLR}$.

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- Replace $\equiv_T$ with $\equiv_{wtt}$? No.
- Replace MLR with SR and $\equiv_T$ with $\equiv_{wtt}$? No.

Why the *wtt* versions fail

The main idea behind each proof:

- Pick a specific sequence $A \in \mathrm{MLR}$ that computes a sufficiently fast-growing function.
- Define a *tt*-functional Φ in terms of this fast-growing function such that $\Phi(A)$ spreads out the randomness in A.
- Show that $\Phi(A)$ has low initial segment complexity.
- Show that sequences that *wtt*-compute some $B \in \mathrm{MLR}$ must have high initial segment complexity.

An interesting consequence

Theorem (Bienvenu, Porter)

Given a Turing degree $\mathbf{a}$ *containing some* $A \in \mathrm{MLR}$*, there is some* $B \in \mathbf{a}$ *such that*

$$B \in \mathrm{MLR}_{comp} \cap \mathrm{NCR}_{comp}$$

if and only if $\mathbf{a}$ *is hyperimmune.*

$$\mathrm{MLR}_{comp} = \{X \in 2^\omega : X \in \mathrm{MLR}_\mu \text{ for some computable } \mu\}.$$

$$\mathrm{NCR}_{comp} = \{X \in 2^\omega : X \notin \mathrm{MLR}_\mu \text{ for any comp., cont. } \mu\}.$$

Hereafter, let us refer to sequences in MLR_{comp} as *proper* sequences.

Some questions

These results raise some questions that will occupy us today:

1. How does the initial segment complexity of a proper sequence relate to its ability to compute fast-growing functions?
2. How does the initial segment complexity of a proper sequence reflect properties of the underlying measure with respect to which it is random?
3. How do we reconcile the fact that some proper sequences have low initial segment complexity with the Levin-Schnorr theorem, which appears to tell us that the initial segment complexity of a proper sequence is high?

Outline of the remainder of the talk

1. Basics of initial segment complexity
2. Random sequences with high initial segment complexity
3. Random sequences with low initial segment complexity

1. Basics of Initial Segment Complexity

Kolmogorov complexity

Let $U : 2^{<\omega} \to 2^{<\omega}$ be a universal, prefix-free Turing machine.

For each $\sigma \in 2^{<\omega}$, the *prefix-free Kolmogorov complexity* of σ is defined to be

$$K(\sigma) := \min\{|\tau| : U(\tau)\downarrow = \sigma\}.$$

The Levin-Schnorr Theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ *is Martin-Löf random if and only if*

$$\forall n \, K(X \restriction n) \geq n - O(1).$$

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$X \in 2^\omega$ is Martin-Löf random if and only if

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More generally, we have the following:

The Levin-Schnorr Theorem

Theorem (Levin, Schnorr)

$X \in 2^\omega$ is Martin-Löf random if and only if

$$\forall n\, K(X \restriction n) \geq n - O(1).$$

More generally, we have the following:

Theorem

Let μ be a computable measure. $X \in 2^\omega$ is μ-Martin-Löf random if and only if

$$\forall n\, K(X \restriction n) \geq -\log(\mu(X \restriction n)) - O(1).$$

A priori complexity

Definition

- A *semi-measure* is a function $\rho : 2^{<\omega} \to [0,1]$ satisfying
  (i) $\rho(\epsilon) = 1$ and
  (ii) $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.
- A semi-measure ρ is *left-c.e.* if ρ is computably approximable from below.

Fact: There exists a *universal* left-c.e. semi-measure M. That is, for every left-c.e. semi-measure ρ there is some c such that

$$c \cdot M(\sigma) \geq \rho(\sigma)$$

for every σ.

We define the *a priori complexity* of $\sigma \in 2^{<\omega}$ to be

$$KA(\sigma) := -\log M(\sigma).$$

Complex and strongly complex sequences

Recall that an order function $h : \omega \to \omega$ is an unbounded, non-decreasing function.

Definition

Let $X \in 2^\omega$.

- X is *complex* if there is a computable order function $h : \omega \to \omega$ such that

$$\forall n \; K(X{\upharpoonright}n) \geq h(n).$$

- X is *strongly complex* if there is a computable order function $g : \omega \to \omega$ such that

$$\forall n \; KA(X{\upharpoonright}n) \geq g(n).$$

Proposition

X is complex if and only if X is strongly complex.

2. Random sequences with high initial segment complexity

What counts as high initial segment complexity?

In what follows, we will consider a proper sequence to have high initial segment complexity if it is complex.

It is worth noting that not every complex sequence is proper.

For example, there is a complex sequence of minimal Turing degree, but no proper sequence has minimal Turing degree.

A preliminary observation

Suppose that X is Martin-Löf random with respect to a computable measure μ.

Then by the Levin-Schnorr theorem,

$$\forall n \; K(X\restriction n) \geq -\log(\mu(X\restriction n)) - O(1).$$

Note that this does not imply that X is complex, since the function $n \mapsto -\log(\mu(X\restriction n))$ is in most cases not computable but only X-computable.

A sufficient condition for complexity

Theorem (Hölzl, Merkle, Porter)

If $X \in 2^\omega$ is Martin-Löf random with respect to a computable, continuous measure μ, then X is complex.

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Theorem (Hölzl, Merkle, Porter)

If $X \in 2^\omega$ is Martin-Löf random with respect to a computable, continuous measure μ, then X is complex.

This follows from the following two results.

Lemma

Let μ be a computable, continuous measure and let $X \in \mathrm{MLR}_\mu$. Then there is some Martin-Löf random $Y \leq_{tt} X$.

Lemma

If Y is complex and $Y \leq_{wtt} X$, then X is complex.

What about the converse?

The converse of the previous theorem doesn't hold: as stated earlier, there are complex sequences that are not proper.

However, we do have a partial converse.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

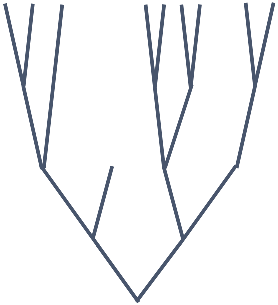
A useful lemma

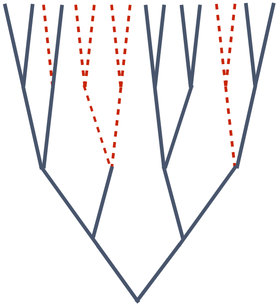
Lemma

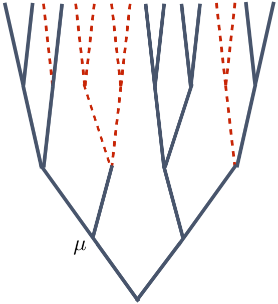
Suppose that

- μ *is a computable measure,*
- $X \in \mathrm{MLR}_\mu$ *is non-computable,*
- $\mathcal{P}$ *is a* Π^0_1 *class with no computable members, and*
- $X \in \mathcal{P}$.

Then there is some computable, continuous measure ν *such that* $X \in \mathrm{MLR}_\nu$.



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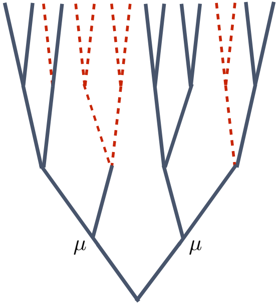
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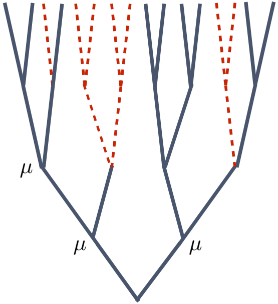

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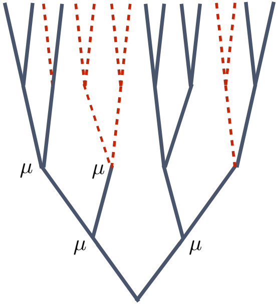

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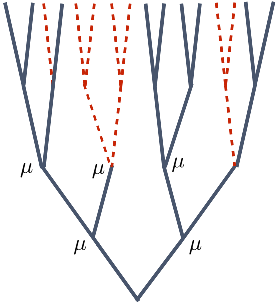
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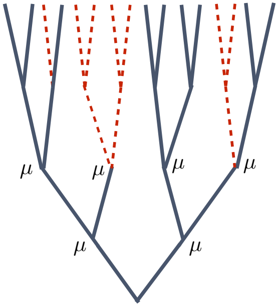
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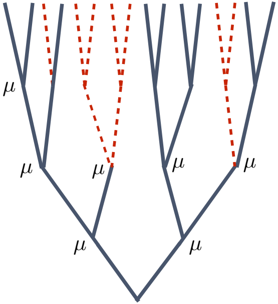
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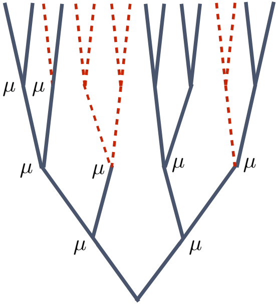
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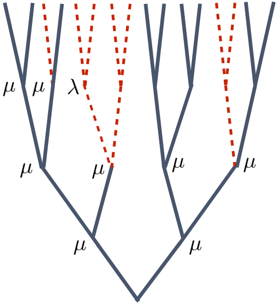

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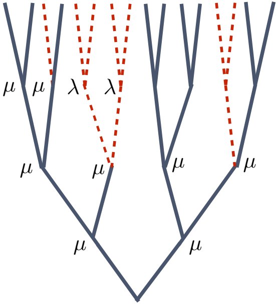
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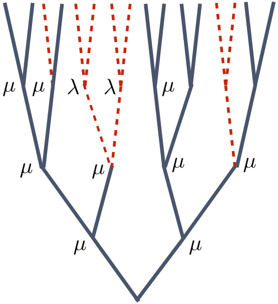

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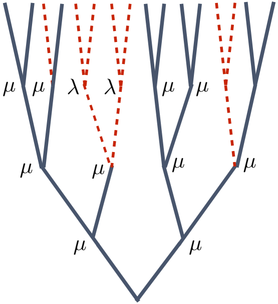

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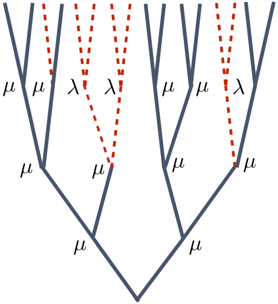
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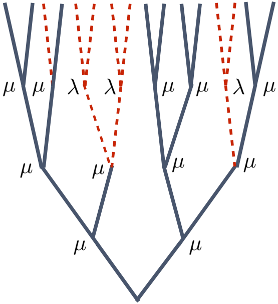
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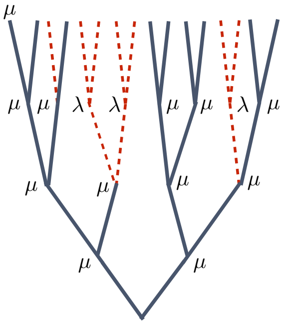
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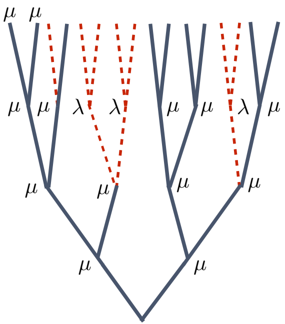
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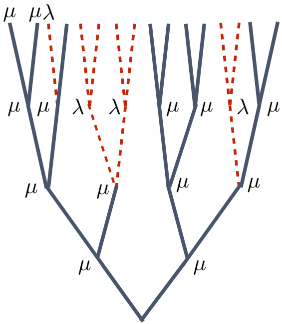

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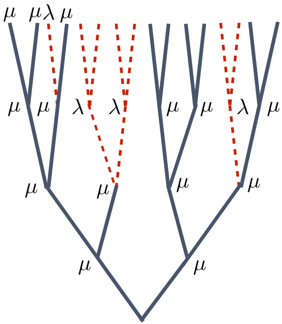

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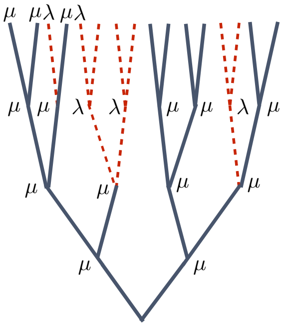
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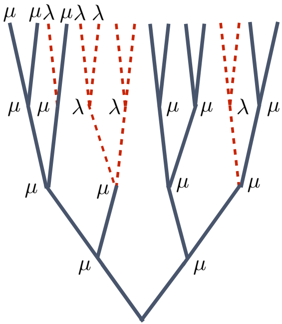
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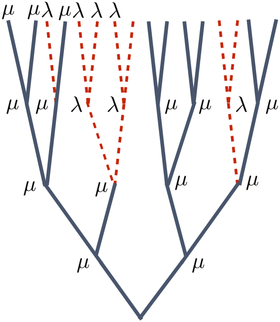
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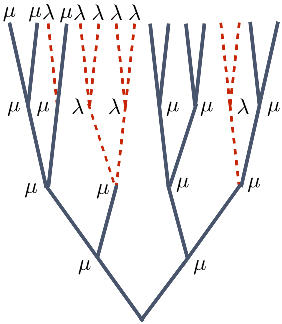
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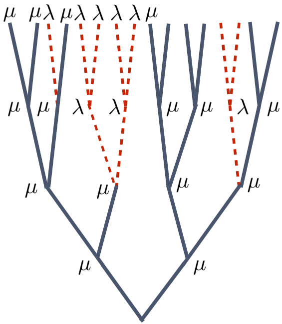
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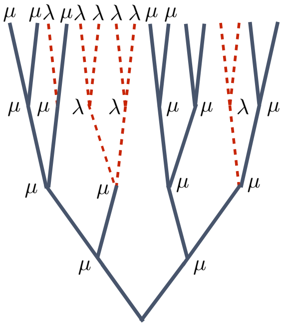
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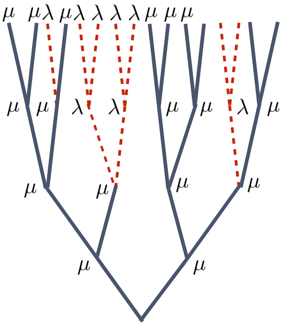
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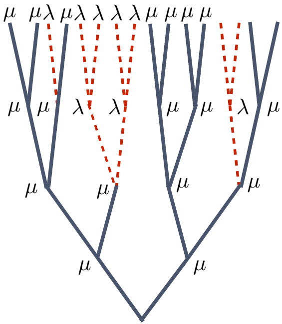
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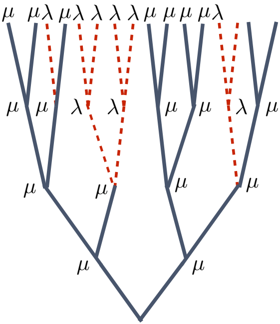
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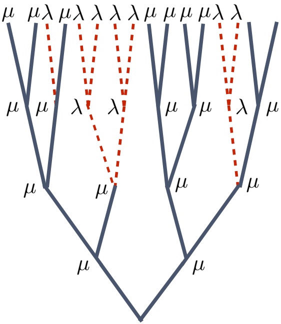
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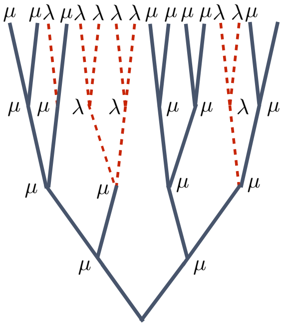
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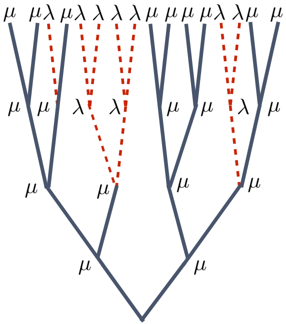
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Establishing the partial converse

Theorem

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

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Theorem

Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.

To prove this theorem, let h be the computable order function that witnesses that X is complex.

Then we apply the previous lemma to the Π^0_1 class

$$\{A \in 2^\omega : (\forall n) K(A\restriction n) \geq h(n)\},$$

which contains X but no computable sequences.

Connection to semigenericity

Definition

$X \in 2^\omega$ is *semigeneric* if for every Π^0_1 class $\mathcal{P}$ containing X contains some computable member.

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. $X \notin \mathrm{NCR}_{comp}$.
2. *X is complex.*
3. *X is not semigeneric.*

Avoidability and hyperavoidability

Definition

(i) $X \in 2^\omega$ is *avoidable* if there is some partial computable function p, called an *avoidance function*, such that for every computable set M and every index e for M, $p(e)\downarrow$ and $X \upharpoonright p(e) \neq M \upharpoonright p(e)$.

(ii) Moreover, X is *hyperavoidable* if X is avoidable with a total avoidance function.

- Not every avoidable sequence is hyperavoidable.
- X is hyperavoidable if and only if X is complex.
- A non-computable sequence X is avoidable if and only if X is not semigeneric.

Additional consequences

Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. $X \in \mathrm{MLR}_\mu$ *for some computable, continuous* μ.
2. X *is complex.*
3. X *is not semigeneric.*

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Theorem (Hölzl, Merkle, Porter)

Let $X \in 2^\omega$ be proper. The following are equivalent:

1. $X \in \mathrm{MLR}_\mu$ *for some computable, continuous* μ.
2. X *is complex.*
3. X *is not semigeneric.*
4. X *is hyperavoidable.*
5. X *is avoidable.*

A follow-up question

Let μ be a computable, continuous measure.

Since every sequence that is random with respect μ is complex, is there a single computable order function that witnesses the complexity of μ-random sequences?

Is there a least such function (up to an additive constant)?

A follow-up result

Definition

Let μ be a continuous measure. Then the *granularity function of μ*, denoted g_μ, is the order function mapping n to the least ℓ such that $\mu(\sigma) < 2^{-n}$ for every σ of length ℓ.

A follow-up result

Definition

Let μ be a continuous measure. Then the *granularity function of μ*, denoted g_μ, is the order function mapping n to the least ℓ such that $\mu(\sigma) < 2^{-n}$ for every σ of length ℓ.

Theorem (Hölzl, Merkle, Porter)

Let μ be a computable, continuous measure and let $X \in \mathrm{MLR}_\mu$. Then we have

$$\forall n \; KA(X \upharpoonright n) \geq g_\mu^{-1}(n) - O(1).$$

Some facts about the granularity of a computable measure

- If μ is exactly computable, that is, μ is $\mathbb{Q}_2$-valued and the function $\sigma \mapsto \mu(\sigma)$ is a computable function, then g_μ is computable.

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Some facts about the granularity of a computable measure

- If μ is exactly computable, that is, μ is $\mathbb{Q}_2$-valued and the function $\sigma \mapsto \mu(\sigma)$ is a computable function, then g_μ is computable.
- However, there is a computable, continuous measure μ such that the granularity function g_μ of μ is not computable.
- For every computable, continuous measure μ, there is a computable order function $f : \omega \to \omega$ such that

$$|f(n) - g_\mu(n)^{-1}| \leq O(1).$$

Such a function f provides as a global computable lower bound for the initial segment complexity of every μ-random sequence.

A question about uniformity

Question

If we have a computable, atomic measure μ such that

$$\forall X \in 2^\omega \; (X \in \mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \;\Rightarrow\; X \text{ is complex}),$$

is there a computable, continuous measure ν such that

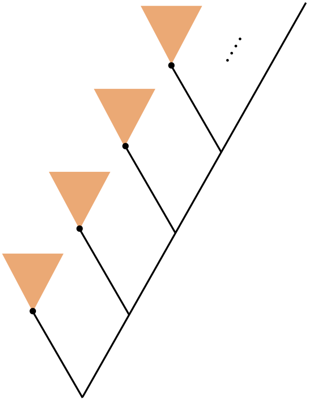
$$\mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \subseteq \mathsf{MLR}_\nu?$$

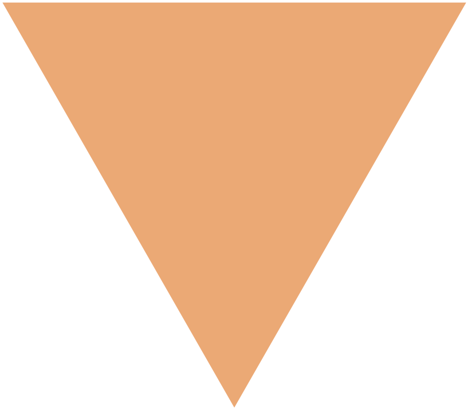
An answer

Theorem (Hölzl, Merkle, Porter)

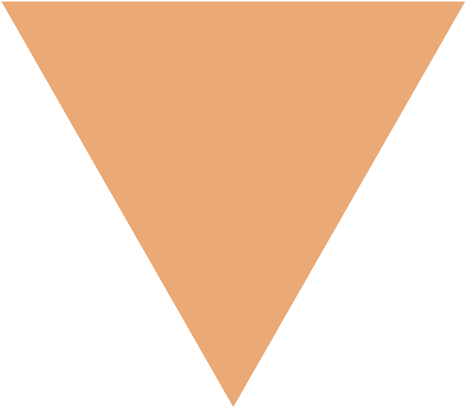
There is a computable, atomic measure μ such that

- *every $X \in \mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu$ is complex but*
- *there is no computable, continuous measure ν such that $\mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \subseteq \mathsf{MLR}_\nu$.*





the i^{th} neighborhood



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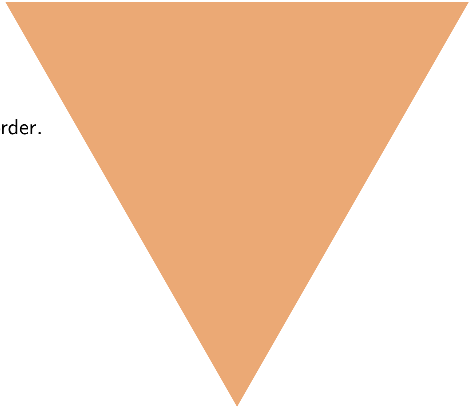

Suppose that ϕ_i is an order.

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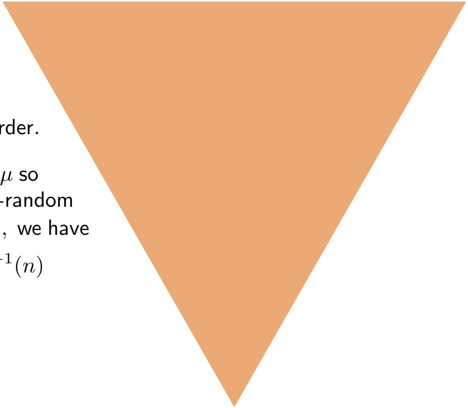

Suppose that ϕ_i is an order.

We define the measure μ so that for any complex μ-random X in this neighborhood, we have

$$KA(X\restriction n) < \phi_i^{-1}(n)$$

for almost every n.

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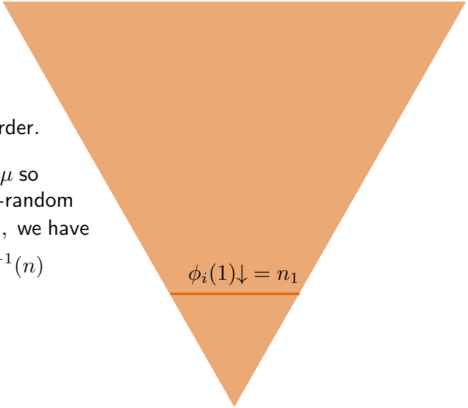

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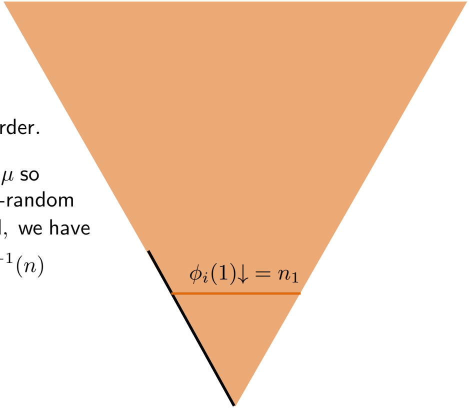

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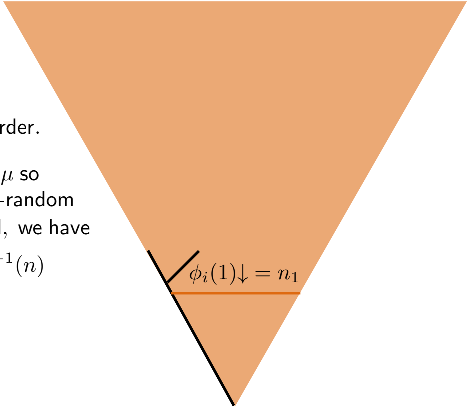

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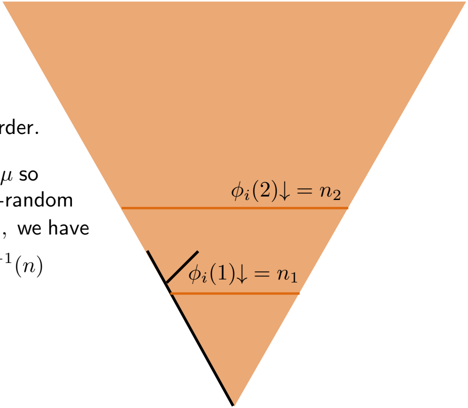

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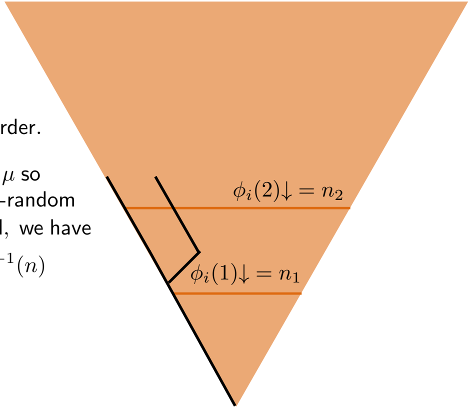

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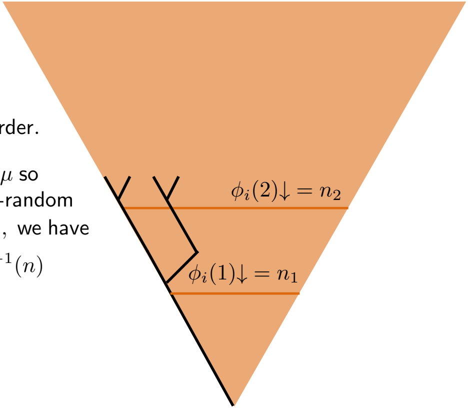

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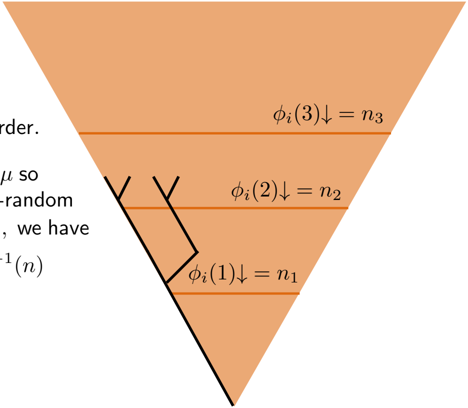

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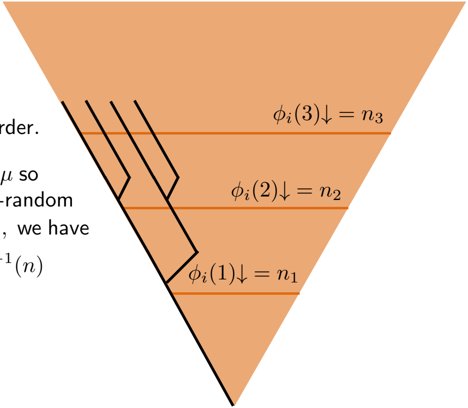

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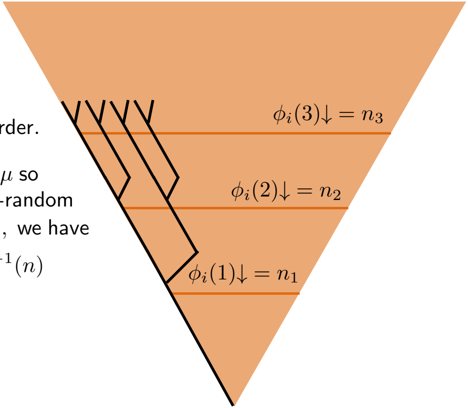

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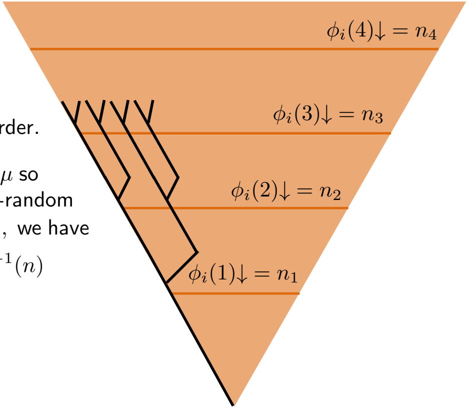

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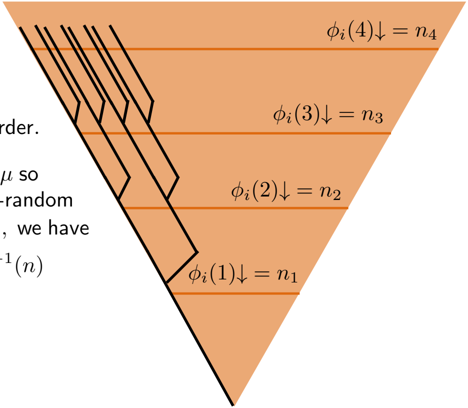

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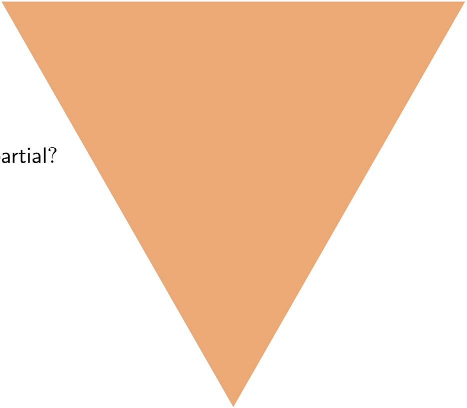

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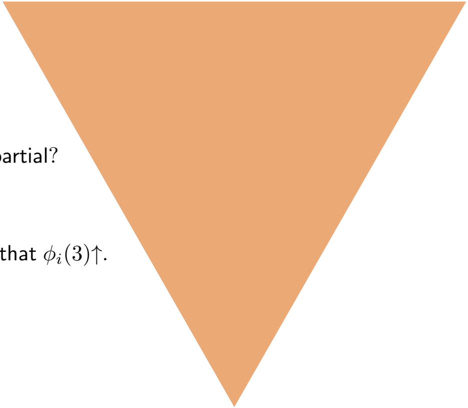

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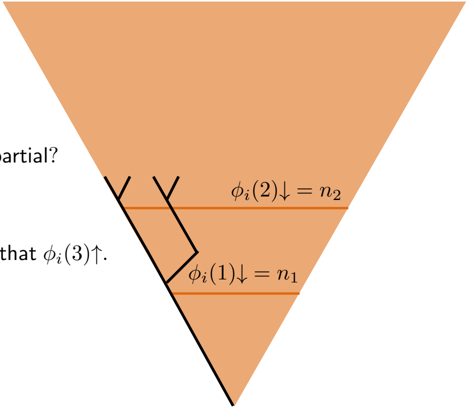

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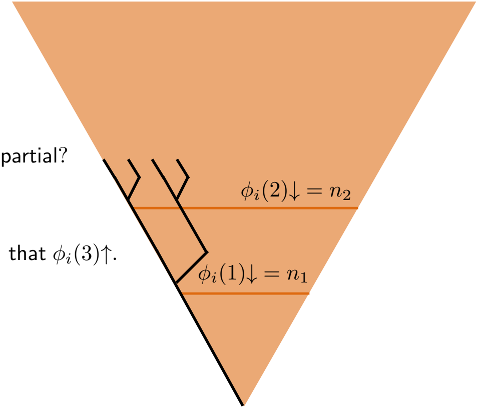

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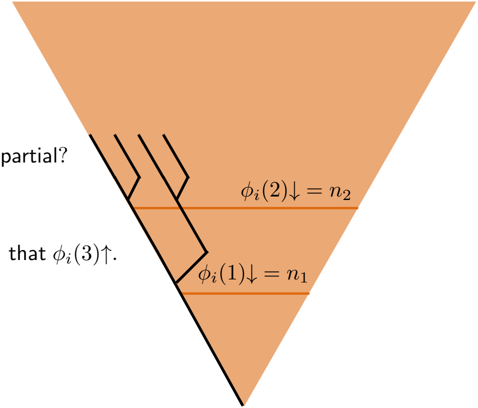

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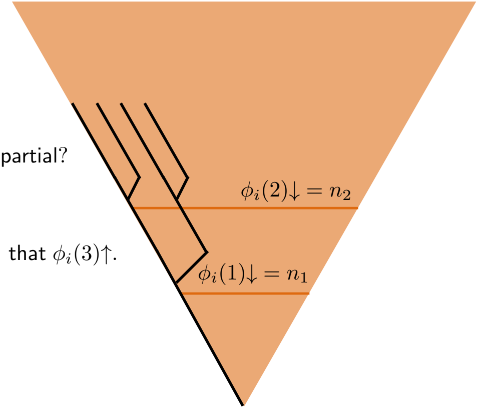

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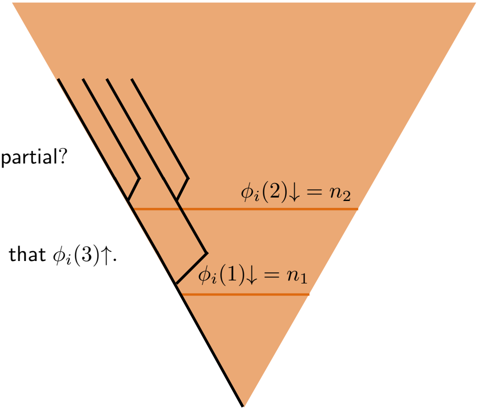

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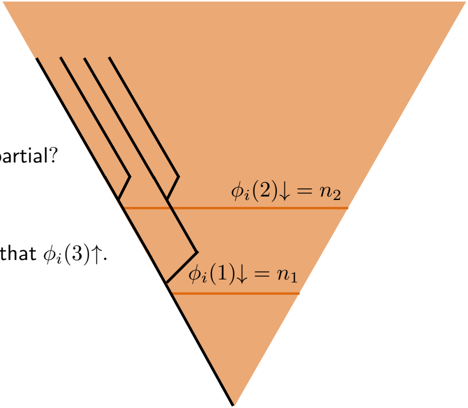

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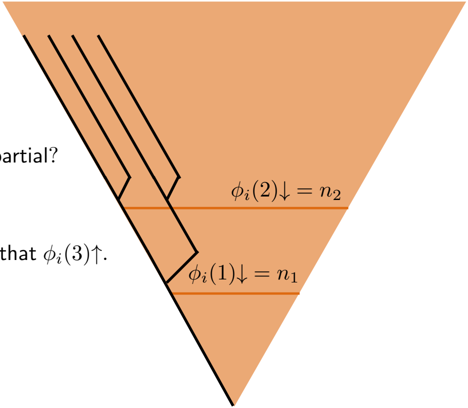

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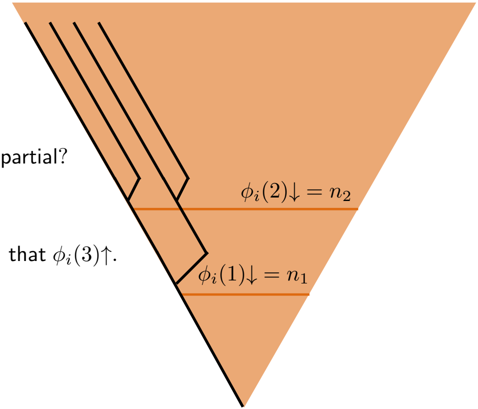

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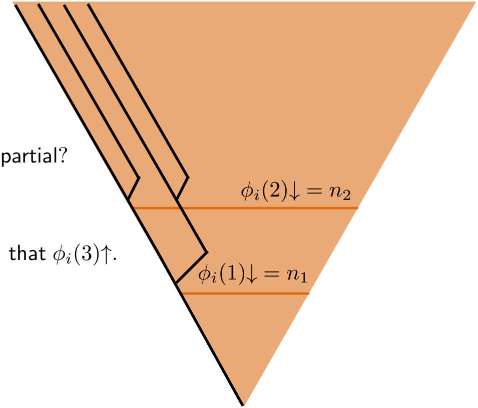

Let $[\![\sigma_i]\!]$ be the i^{th} neighborhood.

One can verify that

- if ϕ_i is partial, then $[\![\sigma_i]\!] \cap \mathsf{MLR}_\mu \subseteq \mathsf{Atoms}_\mu$;
- if ϕ_i is total, then $[\![\sigma_i]\!] \cap \mathsf{Atoms}_\mu = \emptyset$ and every $X \in \mathsf{MLR}_\mu \cap [\![\sigma_i]\!]$ is complex.

Lastly, if there is some computable, continuous ν such that $\mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu \subseteq \mathsf{MLR}_\nu$, then there is a computable order $f = \phi_i$ such that for every $X \in \mathsf{MLR}_\mu \setminus \mathsf{Atoms}_\mu$,

$$KA(X{\upharpoonright}n) \geq f^{-1}(n) - O(1)$$

for every n, which yields a contradiction.

3. Random sequences with low initial segment complexity

Notions of non-complexity

Definition

(i) X is *infinitely often complex* (or *i.o. complex*) if there is some computable order function f such that $K(X \upharpoonright f(n)) \geq n$ for infinitely many n.

(ii) X is *anti-complex* if for every computable order function f we have $K(X \upharpoonright f(n)) \leq n$ for almost every n.

(iii) X is *infinitely often anti-complex* (or *i.o. anti-complex* if for every computable order function f we have $K(X \upharpoonright f(n)) \leq n$ for infinitely every n.

not complex $\Rightarrow$ i.o. anti-complex
not anti-complex $\Rightarrow$ i.o. complex

KA-versions of non-complexity

Each of the notions on the previous slide can equivalently be formulated in terms of a priori complexity (KA).

One potential benefit of working with KA rather than K in this context is given by the following result, which does not hold for K.

Lemma

$X \in 2^\omega$ is anti-complex if and only if for every computable order f, $KA(X \upharpoonright n) \leq f(n) + O(1)$.

Proper non-complex sequences

By our earlier result, if a proper sequence is not random with respect to any continuous, computable measure, it cannot be complex and must be i.o. anti-complex.

We have already seen examples of such sequences:

- The counterexamples to the *wtt*-versions of Demuth's Theorem are proper and non-complex.

I.o. anti-complex proper sequences

In fact, we can recast the theorem from the beginning of the talk:

Theorem (Bienvenu, Porter)

Let **a** *be a random Turing degree. Then* **a** *contains an i.o. anti-complex proper sequence if and only if* **a** *is hyperimmune.*

With some additional work, this can be slightly improved.

Theorem (Hölzl, Merkle, Porter)

Let **a** *be a random Turing degree. Then* **a** *contains an i.o. anti-complex, i.o. complex proper sequence if and only if* **a** *is hyperimmune.*

Anti-complex proper sequences

We have a similar (though not quite optimal) result for anti-complex proper sequences.

Theorem (Hölzl, Merkle, Porter)

Let **a** *be a random degree.*

(i) *If there is some Martin-Löf random* $A \in \mathbf{a}$ *and a function* $f \leq_{wtt} A$ *that dominates all computable functions, then there is some anti-complex, proper sequence* $B \equiv_T A$.

(ii) *If* **a** *contains an anti-complex, proper sequence, then* **a** *is high.*

Question

If **a** is high and random, does **a** contain an anti-complex, proper sequence? That is, can we replace the $\leq_{wtt}$ in (i) with $\leq_T$?

Thank you!